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ACOUSTICS OF A NONHOMOGENEOUS MOVING MEDIUM

By D. I. Blokhintsev

Translation

"Akustika Neodnorodnoi Dvizhushcheisya Sredy." Ogiz, Gosudarstvennoe Izdatel'stvo, Tekhniko-Teoreticheskoi Literatury, Moskva, 1946, Leningrad.

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PREFACE

The practical problems brought about by the Great War have given rise to theoretical problems.

In acoustics interest centers about the problem of the propagation of sound in a nonhomogeneous moving medium, which is the nature of the atmosphere and the water of seas and rivers, as well as about problems concerning moving sources and sound receivers. These problems are closely connected; they lie at the boundary between acoustics and hydrodynamics in the broad sense of the word.

It is precisely these aspects of acoustics that have been either little developed theoretically and experimentally or are not very popular among acoustics technicians. This is the circumstance that has provided the occasion for the appearance of this book, which is devoted to the theoretical basis of the acoustics of a moving nonhomogeneous medium. Experiments are considered only to illustrate or confirm some theoretical explanation or derivation.

As regards the choice of theoretical questions and their treatment, the book does not in any way pretend to be complete. The choice of material was to a considerable extent dictated by the author's own investigations, some of which were, previously published and others first presented herein. Certain problems were not worked through to the end but have merely been indicated. The author, nevertheless, included them in the book, on account of the creative interest which they may arouse among investigators in the field of theoretical acoustics. The author expresses his appreciation to N. N. Andreev and S. I. Rzhevkin, who were acquainted with the manuscript of this book, for their useful advice and comments, and also to L. D. Landau, whose consultation made possible the clarification of a number of problems.

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CHAPTER I

ACOUSTICS EQUATIONS OF A NONHOMOGENEOUS MOVING MEDIUM

1. Outline of Dynamics of a Compressible Fluid

The medium in which sound is propagated, whether it is a gas, a liquid, or a solid body, has an atomic structure. If, however, the frequency of the sound vibrations is not too large, this atomic character of the medium may be ignored.

For a gas it may be shown (ref. 1) that if $f \ll 1/\tau$, where f is the frequency of the vibrations and τ the time taken to traverse the free path between collisions, the gas may be considered as a dense medium characterized by certain constants. This method of considering the problem is assumed in aerodynamics and in the theory of elasticity. Since the atomic character of the medium is ignored, the phenomenon of the dispersion of sound cannot, in all strictness, be taken into account. Fortunately, in the majority of practical problems, the dispersion of sound does not have great significance. For this reason, phenomena which require consideration of the atomic nature of the medium will not be considered, and the aerodynamic equations of a compressible gas will be used as the basis of the theoretical analysis of the acoustics of a moving medium.

These equations are first considered without the assumption of any specific restrictions for the acoustics (such as large frequency and small amplitude of vibrations). The equations of the dynamics of a compressible gas express the three fundamental laws of conservation: (1) conservation of matter, (2) conservation of momentum, and (3) conservation of energy. In order to formulate these laws, a certain system of coordinates x, y, and z, fixed relative to the undisturbed medium, is chosen. Further, t is the time, $\vec{v}$ is the velocity of the gas in this system (Translator's note: An arrow is used in the typescript to indicate that a symbol stands for a vector), $v_1 = v_x$, $v_2 = v_y$, and $v_3 = v_z$ are the components of $\vec{v}$ along the x, y, and z axes, respectively, and ρ is the density of the gas. In these notations, the law of the conservation of matter, mathematically expressed by the equation of continuity, assumes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho v_k) = 0 \tag{1.1}$$

where the summation is carried out for $k = 1, 2$, and 3. The vector $\rho\vec{v}$ is the flow density vector of the substance. This equation states that the change in amount of substance in any small volume is equal to the flow of the substance through the surface enclosing this volume.

The vector $\rho\vec{v}$ may be considered also as the vector of the momentum density. The change of momentum in any small volume should be equal to the momentum transported by the motion of the fluid through the surface enclosing this volume plus the force applied to the volume.

The momentum flow due to the transport of momentum is a tensor with the components: $\rho v_i v_k$ $(i,k = 1,2,3)$. The assumption is made that there are no volume forces. Hence the force applied to the volume is equal to the resultant of the stresses applied to the surface of the volume. The tensor of these stresses will be denoted by T_{ik} and is composed of the scalar pressure p and the viscous components s_{ik}

$$T_{ik} = p \cdot \delta_{ik} - s_{ik} \tag{1.2}$$

where $\delta_{ik} = 1$ if $i = k$, and $\delta_{ik} = 0$ if $i \neq k$.

When applied to a small volume, the law of the conservation of momentum can be written in the form

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_k}(T_{ik} + \rho v_i v_k) = 0 \tag{1.3}$$

i and $k = 1, 2$, and 3 and again is summed for $k = 1, 2$, and 3. The equation of the conservation of energy should express the fact that the change in the total energy in a small volume, made up of the kinetic energy and the internal energy of a unit volume of the gas, is equal to the flow of the kinetic and internal energy through the surface enclosing this volume, the heat flow through this surface plus the work performed by the stresses acting on this volume. The part of the energy flow vector due to the transport of the kinetic energy $\rho \cdot \frac{v^2}{2}$ and the internal energy ρE (E is the energy of unit mass of the gas) is $\left(\rho \frac{v^2}{2} + \rho E\right)\vec{v}$. If the heat flow vector is denoted by $\vec{S}(S_1,S_2,S_3)$ and the conservation law is applied to a small volume,

$$\frac{\partial}{\partial t}\left(\rho \frac{v^2}{2} + \rho E\right) + \frac{\partial}{\partial x_k}\left[\left(\rho \frac{v^2}{2} + \rho E\right) v_k + S_k\right] + \frac{\partial}{\partial x_k}(v_i T_{ik}) = 0 \tag{1.4}$$

where the summation is for i and $k = 1, 2,$ and 3. The last term gives the work of the stresses on a unit volume. For an isotropic, homogeneous liquid (or gas), the stresses S_{ik} are connected with the deformations v_{ik} according to the Newtonian relation[1]

$$S_{ii} = 2\mu v_{ii} + \gamma \cdot \operatorname{div} \vec{v}; \quad S_{ik} = 2\mu \cdot v_{ik} \tag{1.5}$$

where μ is the viscosity of the gas and v_{ik} is the tensor of the deformations

$$v_{ik} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) \tag{1.6}$$

The magnitude γ can be written in the form $\gamma = \mu' - 2\mu/3$, where μ' is the so-called second coefficient of viscosity (see ref. (1)). With this coefficient, account is taken of the conversion of the energy of the macroscopic motion of a gas into the energy of the internal degrees of freedom of the molecules (the rotation of the molecules), a fact which is of appreciable significance only for ultrasonic frequencies. For this reason, in the majority of cases the assumption may be made that $\mu' = 0$ and $\gamma = -2\mu/3$ (the value assumed in the theory of Stokes).

The flow of heat $\vec{S}$ expressed in terms of the gradient of the absolute temperature T is

$$S_k = \lambda \cdot \frac{\partial T}{\partial x_k}; \quad \lambda = \rho \cdot c_v \varkappa \tag{1.7}$$

where $\varkappa$ is the coefficient of the heat conductivity of the gas and c_v is the specific heat of the gas at constant volume.

To the three fundamental hydrodynamic equations, (1.1), (1.3), and (1.4), the equation of state of the gas (or liquid) connecting the pressure p, the density ρ, and the temperature T is added

$$p = Z(\rho, T) \tag{1.8}$$

Equations (1.1), (1.3), and (1.4) permit a rational determination of the flow of substance $\vec{L}$, the flow of momentum represented by the

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[1]This form for v_{ik} follows from the assumption of the isotropic character and homogeneity of the gas or liquid if a linear relation is assumed between the stress tensor s_{ik} and the deformation tensor v_{ik}.

tensor M_{ik}, and the flow of energy $\vec{N}$, which, like the flow of substance, can be written in vector form. This determination will be such that the divergence of the flow, taken with inverse sign, is equal to the derivative with respect to the time of the density of the corresponding magnitude. In this manner from equation (1.1) for the flow of substance (equal to the flow of momentum) the following is obtained:

$$\vec{L} = \rho\vec{v} \tag{1.9}$$

From equation (1.3), substitution of the value of S_{ik} from equation (1.5), gives the tensor of the momentum flow

$$M_{ii} - \rho v_i^2 + p + \gamma \cdot \operatorname{div} \vec{v} - 2\mu \cdot v_{ii}$$

$$M_{ik} = \rho v_i v_k - 2\mu v_{ik} = M_{ki}; \quad i \neq k \tag{1.10}$$

where, as before, i and $k = 1, 2,$ and 3.

The terms of the form ρv_i^2, $\rho v_i v_k$ give the momentum flow due to the transport of momentum by the motion of the fluid, and the terms containing ρ, μ, and γ give the flow of momentum due to the action of the pressure forces and the viscous stresses.

Finally, from equation (1.4), substitution of S_{ik} from equation (1.5) yields the energy flow

$$\vec{N} = \left(\rho \frac{v^2}{2} + \rho E\right) \vec{v} + \vec{S} + \rho\vec{v} + \mu\left\{\nabla v^2 + \left[\operatorname{rot} \vec{v} \times \vec{v}\right]\right\} + \gamma \cdot \operatorname{div} \vec{v} \cdot \vec{v} \tag{1.11}$$

The first term gives the energy flow due to the transport of energy by the fluid, the second ($\vec{S}$) gives the heat flow, and the term[2] $\rho\vec{v}$ and the terms with μ and γ give the part of the energy flow due to the work of the pressure forces and the viscous stresses.

The fundamental equations can also be written in vector from, by substitution of the value of the tensor T_{ik} from equations (1.2) and (1.5) in equations (1.3) and (1.4). Equation (1.1) may, however, be as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0 \tag{1.12}$$

[2]The vector $\vec{N} = \left(\rho \frac{v^2}{2} + \rho\right)\vec{v}$, representing the flow of energy for an ideal incompressible liquid, is called the N. Umov vector (ref. 3).

If use is made of (1.12) equation (1.3) can be written in the form

$$\rho \frac{d\vec{v}}{dt} = - \nabla p + \mu\Delta\vec{v} + \frac{1}{3}\mu\nabla \operatorname{div} \vec{v} \tag{1.13}$$

where ∇ is the symbol for the gradient and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 = \nabla^2$. The magnitude $d\vec{v}/dt$ is the total derivative of the velocity with respect to time and is equal to

$$\frac{\partial\vec{v}}{\partial t} = \frac{\partial\vec{v}}{\partial t} + (\vec{v},\nabla)\vec{v} = \frac{\partial v}{\partial t} + \nabla \frac{v^2}{2} + \left[\operatorname{rot} \vec{v} \times \vec{v}\right] \tag{1.14}$$

The energy equation (eq. (1.4)), with the aid of equation (1.12), assumes the form

$$\rho \frac{dE}{dt} = \lambda \cdot \Delta T + Q - p \cdot \operatorname{div} \vec{v} \tag{1.15}$$

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + (\vec{v},\nabla)E \tag{1.15'}$$

where Q is the dissipative function

$$Q = \sum_{i,k=1}^{3} S_{ik} \cdot v_{ik} \tag{1.16}$$

If this equation is divided by ρ, it may be interpreted so that a change of energy of unit mass dE/dt is equal to the heat flow $\lambda\Delta T/\rho$, the amount of heat divided by the work of the viscous forces Q/ρ, and the work of the pressure forces $(-\rho \operatorname{div} \vec{v}/\rho)$.

This equation may also be interpreted in terms of thermodynamics. The first law of thermodynamics for unit mass of substance yields

$$dE = TdS - p \cdot dV \tag{1.17}$$

where E is the energy of unit mass; S, its entropy; p, the pressure, and V, the specific volume $(V = 1/\rho)$. Thus

$$\frac{dE}{dt} = T \cdot \frac{dS}{dt} - p \frac{dV}{dt} = T \frac{dS}{dt} + \frac{p}{\rho^2} \frac{d\rho}{dt} \tag{1.18}$$

On the other hand,

$$\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + (\vec{v},\nabla)\ \rho = -\ \rho\ .\ \mathrm{div}\ \vec{v} \tag{1.19}$$

so that

$$p\ .\ \frac{\mathrm{div}\ \vec{v}}{\rho} = -\ \frac{p}{\rho^2}\ .\ \frac{d\rho}{dt} \tag{1.20}$$

For adiabatic processes

$$\frac{dE}{dt} = \frac{p}{\rho^2}\ .\ \frac{d\rho}{dt} \tag{1.21}$$

from which

$$E = \int^{p} \frac{dp}{\rho} - \frac{p}{\rho} \tag{1.22}$$

The magnitude

$$w = E + \frac{p}{\rho} = \int^{p} \frac{dp}{\rho} \tag{1.23}$$

is termed the heat function. If the process is nonadiabatic, equation (1.18) holds. From equations (1.15) and (1.18) the following is obtained:

$$T\ \frac{dS}{dt} = \frac{\lambda}{\rho}\ \Delta T + \frac{Q}{\rho} \tag{1.24}$$

The magnitude $T(dS/dt)$ is the increase of heat of unit mass of the gas, which is determined exclusively by the heat conductivity and the work of the friction forces. If λ and μ are neglected since the effects produced by them in the over-all energy balance are usually small corrections, the following results:

$$\frac{dS}{dt} = \frac{\partial S}{dt} + (\vec{v},\ \nabla S) = 0 \tag{1.25}$$

that is, the adiabatic motion of the fluid. The Bernoulli theorem holds for this motion if it is also irrotational (rot $\vec{v} = 0$).

If

$$\vec{v} = -\ \nabla\Phi \tag{1.26}$$

where Φ is the velocity potential, from equations (1.13) and (1.14)

$$\nabla\left[-\frac{\partial\Phi}{\partial t}+\frac{1}{2}(\nabla\Phi)^2\right] = -\frac{\nabla p}{\rho} \qquad (1.27)$$

and since, on the basis of equation (1.23), $p/\rho = \overline{\nabla}w$, integration of equation (1.27) gives

$$w = \int^p \frac{dp}{\rho} = \frac{\partial\Phi}{\partial t} - \frac{1}{2}(\nabla\Phi)^2 \qquad (1.27')$$

If the compressibility of the fluid is neglected,

$$w = \frac{p}{\rho_0} + \text{constant} \qquad (1.28)$$

so that

$$p = \rho_0\frac{\partial\Phi}{\partial t} - \frac{\rho_0}{2}(\nabla\Phi)^2 + \text{constant} \qquad (1.29)$$

and in the case of steady flows $(\partial\Phi/\partial t = 0)$

$$p = \text{constant} - \frac{\rho_0}{2}(\nabla\Phi)^2 = \text{constant} - \frac{\rho_0 v^2}{2} \qquad (1.30)$$

Because the entropy remains constant during the motion for an ideal fluid $(\lambda = \mu = 0)$ introduction of the variables ρ and S in the equation of state, equation (1.8), in place of the variables ρ and T, is expedient since with such a choice of variables one of the variables (S) remains constant, whereas the temperature T varies even for an ideal fluid (for adiabatic compressions and expansions of the fluid). The following may be written in place of equation (1.8)

$$p = Z'(\rho,S) \qquad (1.8')$$

2. Equations of Acoustics in Absence of Wind

The equations which determine the propagation of sound in a motionless medium can now be considered. The vibrations of the medium are called sonic vibrations or simply sound if the amplitude of the vibrations is so small that it is possible to neglect all the changes in state of the gas in any small volume are produced in it by the transport (convection) of mass, momentum, and energy. This situation is the condition of linearity of the vibrations. Further, these vibrations are

assumed to occur with frequencies in the hearing range (the region of classical acoustics) or near this range (infra and ultra sound). Mathematically the above assumption reduces to the neglect of the terms in the aerodynamic equations of a compressible gas which contain second powers or the products of small magnitudes which determine the deviations of the state of the gas from equilibrium. Where π is the deviation of the pressure from the equilibrium value p_0, p is set equal to $p_0 + \pi$, $\rho = \rho_0 + \delta$ where ρ_0 is the value of the density for $p = p_0$ and $T = T_0$, and finally $\vec{v} = \vec{\xi}$($\vec{\xi}$ is a small velocity). Similarly for the temperature, entropy, and energy,

$$T = T_0 + \theta$$

$$S = S_0 + \sigma$$

$$E = E_0 + \varepsilon$$

In place of equations (1.12) and (1.13), the following is obtained:

$$\rho_0 \cdot \frac{\partial \vec{\xi}}{\partial t} = -\nabla\pi + \mu \cdot \Delta\vec{\xi} + \frac{1}{3}\mu\nabla \operatorname{div} \vec{\xi} \tag{1.31}$$

$$\frac{\partial \delta}{\partial t} + \rho_0 \operatorname{div} \vec{\xi} = 0 \tag{1.32}$$

The equation of state of the gas, for an ideal gas in the variables ρ and T is

$$p = \rho \cdot rT \tag{1.33}$$

where r is the gas constant for unit mass; and in the variables ρ and S

$$p = \rho\gamma \cdot \frac{p_0}{\rho_0^{\gamma}} \cdot e^{\frac{S-S_0}{c_v}} \tag{1.34}$$

where c_v is the specific heat at constant volume ($c_v = r/(\gamma-1)$), and $\gamma = c_p/c_v$ is the ratio of the specific heats at constant pressure and constant volume. For small changes of state the following is obtained from equation (1.34):

$$\pi = \gamma \frac{p_0}{\rho_0} \delta + \frac{p_0}{c_v} \sigma + \cdots = c^2\delta + h\sigma + \cdots; \; h = \frac{p_0}{c_v}$$

For $\sigma = 0$, only the first term representing small changes in pressure for small adiabatic compression or expansion of the gas remains. The magnitude

$$c = \sqrt{\gamma \frac{p_0}{\rho_0}} \tag{1.35}$$

is the adiabatic velocity of sound. The second term gives the change in pressure produced by the addition or decrease of heat. The changes of entropy σ obey equation (1.24) which is written by neglecting magnitudes of the second order of smallness as follows:

$$T_0 \frac{\partial \sigma}{\partial t} = \frac{\lambda}{\rho_0} \Delta\theta; \quad \lambda = \rho c_v \varkappa \tag{1.36}$$

The changes in temperature θ may be expressed in terms of the changes in density and entropy. From equation (1.17)

$$T = \left(\frac{\partial E}{\partial S}\right)_\rho \tag{1.37}$$

The energy of an ideal gas is equal to

$$E = c_v T = \frac{p}{(\gamma - 1)\rho} = \frac{p_0}{\rho_0 \gamma} \cdot \frac{\rho^\gamma}{\gamma - 1} \cdot e^{\sigma' c_v} \tag{1.38}$$

from which $\partial E/\partial S = \partial E/\partial \sigma$ is obtained in the form

$$T = \frac{p_0}{\rho_0^\gamma} \cdot \frac{\rho^{\gamma-1}}{(\gamma - 1)c_v} \cdot e^{\sigma' c_v} = \frac{p}{(\gamma - 1)\rho c_v} \tag{1.37'}$$

that is, for small values of ρ and S

$$\theta = \frac{p_0}{\rho_0^2 c_v} \delta + \frac{p_0}{\rho_0(\gamma - 1)c_v} \sigma + \cdots \tag{1.39}$$

where the first term represents the change in temperature during adiabatic compression or expansion of the gas and the second term represents the change in temperature due to the change in entropy of the gas.

Substitution in equation (1.36) yields

$$\frac{\partial \sigma}{\partial t} = \varkappa \Delta\sigma + \varkappa_1 \Delta\delta; \quad \varkappa_1 = \varkappa \frac{(\gamma - 1)c_v}{\rho_0} \tag{1.40}$$

Equations (1.31), (1.32), and (1.40) together with the equation of state (1.34) determine the propagation of sound in a motionless medium when account is taken of the viscosity and heat conductivity of the medium.

The effects arising from the presence of viscosity and heat conductivity reduce, in a first approximation, to the absorption of the sound by the medium. This absorption is generally not large and its magnitude for a plane wave can be determined without difficulty. If its direction of propagation is along the ox axis, the frequency of the sound equals ω, and the wave number vector is equal to k,

$$\left.\begin{aligned} \xi &= \xi_0 e^{i(\omega t - k\cdot x)} \\ \delta &= \delta_0 e^{i(\omega t - k\cdot x)} \\ \sigma &= \sigma_0 e^{i(\omega t - k\cdot x)} \end{aligned}\right\} \tag{1.41}$$

where ξ_0, δ_0, σ_0 are the amplitudes of vibration of the corresponding magnitudes. Substitution of equations (1.41) in equations (1.31), (1.32), and (1.40) yields

$$i\omega\rho_0\xi_0 = ik(c^2\delta_0 + h\sigma_0) - \frac{4}{3}\mu k^2\xi_0 \tag{1.31'}$$

$$i\omega\delta_0 - ik\rho_0\xi_0 = 0 \tag{1.32'}$$

$$i\omega\sigma_0 = -\varkappa k^2\sigma_0 - \varkappa_1 k^2\delta_0 \tag{1.40'}$$

Elimination of the amplitudes gives the relation between k and ω

$$\omega\rho_0 = k\cdot\left[c^2\cdot\frac{k\rho_0}{\omega} - \frac{h\cdot\varkappa_1 k^2}{(i\omega + \varkappa k^2)}\right] + \frac{4}{3}i\mu k^2 \tag{1.42}$$

If k is set equal to $\omega/c - i\alpha$, where α is the coefficient of damping of the wave, the velocity of propagation C' in the first approximation is equal to C, and the damping coefficient α is equal to

$$\alpha = \frac{2}{3}\frac{\mu\cdot\omega^2}{\rho c^3} + \frac{\varkappa}{2\rho}\left(1 - \frac{a^2}{c^2}\right)\frac{\omega^2}{c^3} \tag{1.43}$$

where $a^2 = p_0/\rho_0$ is the square of the isothermal velocity of sound. For air $\alpha = 1{,}1 \cdot 10^{-13}f^2\text{cm}^{-1}$, where $f = \omega/2\pi$ is the frequency of sound in Hz (1 Hertz = 1 cycle/sec). Hence in many cases the effect of the viscosity and heat conductivity may be neglected or their effect taken into account by introduction of the absorption coefficient in the final results. The smallness of the effect of viscosity and heat conductivity of the air on the propagation of sound is determined not only by the smallness of the coefficients μ and $\varkappa$ but also by the smallness of the gradients of all magnitudes which vary in the sound propagation.

Equations (1.31) and (1.40) show that these gradients enter the equation in the form of second derivatives of ξ, σ, and so forth (for example, $\mu\Delta\xi$ and $\varkappa\Delta\sigma$). In the propagation of a wave in free space these derivatives are in order of magnitude equal to ξ/λ^2, σ/λ^2, $\cdots$, and so forth, and become appreciable only for very short wave lengths (as the final equation for the absorption coefficient α shows since α increases proportionally to the square of the frequency.

Near the boundaries of solid or fluid bodies which may be considered as stationary, the losses by viscosity and heat conductivity increase. In these cases sharper changes of state of the gas in space occur and the second derivatives of ξ, σ, and δ are determined not by the length of the wave but either by the dimensions of the body l so that $\Delta\xi \cong \xi/l^2$ and $\Delta\sigma \cong \sigma/l^2$ or by the "natural" length $d' = \sqrt{\nu/\omega}$ (this length is in addition to the lengths λ and l, and is determined from dimensional considerations), where ν is the kinematic viscosity ($\nu = \mu/\rho$), or by the length $d'' = \sqrt{\varkappa/\omega}$. In these cases the order of the magnitudes is given by $\Delta\xi \cong \xi/d^2$ and $\Delta\sigma \cong \sigma/d^2$.

In general, the losses by viscosity and heat conductivity near the boundary of a solid or fluid body are determined by the least of the three lengths λ, l, and $d(d', d'')$.

Despite the increase in the losses near walls and stationary boundaries, the losses remain small and can be considered a correction to the motion which occurs without losses (except for the case of the propagation of sound in very narrow channels). An example of the approximate computation of the effects of viscosity and heat conductivity may be found in the work of the author (ref. 4).

In addition to the absorption of sound associated with the heat conductivity and the viscosity of the medium still another molecular absorption of sound exists which was discovered by V. Knudsen (ref. 5) and explained by G. Kneser (ref. 6). The physical character of this absorption lies in the conversion of the energy of the sound vibrations into the energy of inner molecular motion (energy of rotation of the

molecules). This absorption likewise increases with the frequency and is of special significance for the ultrasonic range.

As the consideration of these problems deviates from the present subject, discussion is limited to the references given.

In all those cases where the losses of the sound energy are not of interest, the viscosity and heat conductivity of the air may be ignored. If λ and μ are set equal to 0 in equations (1.3') and (1.40), $\sigma = 0$, that is, adiabatic propagation of sound is obtained and the equations describing this propagation assume the form

$$\rho_0 \cdot \frac{\partial \vec{\xi}}{\partial t} = - \nabla \pi \tag{1.44}$$

$$\frac{\partial \delta}{\partial t} + \rho_0 \cdot \operatorname{div} \vec{\xi} = 0 \tag{1.45}$$

$$\pi = c^2 \delta \tag{1.46}$$

These equations may be solved with the aid of the single function φ which is termed the velocity potential (or simply the potential). The first three equations (1.44) are satisfied by setting

$$\left.\begin{aligned} \pi &= \rho_0 \cdot \frac{\partial \varphi}{\partial t} \\ \vec{\xi} &= - \nabla \varphi \end{aligned}\right\} \tag{1.47}$$

The wave equation for the potential from equations (1.46) and (1.45) is obtained:

$$\Delta \varphi - \frac{1}{c^2} \cdot \frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{1.48}$$

which, in the presence of bodies, must be solved with the boundary condition

$$- \left(\frac{\partial \varphi}{\partial n} \right) = \xi_{0n} \quad \text{(on the surface of the body)} \tag{1.49}$$

where $\partial / \partial n$ is the derivative along the normal to the surface of the body and ξ_{0n} is the normal velocity of the surface of the body assumed as small. In place of equation (1.49), for stationary bodies

$$\frac{\partial \varphi}{\partial n} = 0 \quad \text{(on the surface of the body)} \tag{1.49'}$$

For a unique solution of the problem of the sonic field described by equation (1.48) the initial conditions for φ and $\partial\varphi/\partial t$ must be formulated in addition to the boundary conditions of equations (1.49) or (1.49').

3. Energy and Energy Flow in Acoustics

For linear acoustics all magnitudes referring to the sound are computed with an accuracy up to the first degree of the amplitude A, which may, for example, be the amplitude of a piston which excites sound vibrations. Achievement of more accurate solutions of the equations of hydrodynamics will yield the succeeding approximation containing terms proportional to A^2, and so forth (when account is taken of nonlinear phenomena). For the pressure p, the density ρ, and the velocity of motion $\vec{v}$, the following series is written:

$$p = p_0 + \pi_1 + \pi_2 + \cdots$$

$$\rho = \rho_0 + \delta_1 + \delta_2 + \cdots$$

$$\vec{v} = \vec{v}_0 + \vec{\xi}_1 + \vec{\xi}_2 + \cdots \tag{1.50}$$

The magnitudes p_0, ρ_0, and $\vec{v}_0$ refer to the motion undisturbed by the sound; the magnitudes π_1, δ_1, and $\vec{\xi}_1$ are proportional to A, the magnitudes π_2, δ_2, and ξ_2 are proportional to A^2, and so forth. The energy and energy flow contain the squares of the magnitudes δ_1, $\vec{\xi}_1$, and π_1. For this reason caution must be used when the energy and energy flow are computed in linear acoustics, as was pointed out by I. Bronshtein and B. Konstantinov (ref. 7) and also by N. N. Andreev (ref. 8), since these magnitudes, being of the order of A^2, may also contain the first degrees of the succeeding approximations π_2, δ_2, and $\vec{\xi}_2$ while their contribution will be of the same order as the contribution from the squares of π_1, δ_1, and $\vec{\xi}_1$.

The general expression for the energy density of a compressible medium is

$$U = \frac{\rho v^2}{2} + \rho E \tag{1.51}$$

where E is the internal energy of unit mass of the medium. The energy flow $\vec{N}$, computed on the basis of equation (1.11) with the viscosity and heat conductivity neglected, is equal to

$$\vec{N} = U\vec{v} + p\vec{v} \tag{1.52}$$

From the law of the conservation of energy,

$$\frac{\partial U}{\partial t} + \mathrm{div}\,\vec{N} = 0 \tag{1.53}$$

This equation is one of the fundamental equations of hydrodynamics, that is, equation (1.4) for the case of an ideal fluid ($\mu = \lambda = \gamma = 0$).

For an ideal gas $\rho E = p/(\gamma - 1)$ (equation (1.38)); hence

$$\vec{N} = \frac{\rho v^2}{2}\vec{v} + \frac{\gamma p\vec{v}}{\gamma - 1} \tag{1.52'}$$

For acoustics the initial medium is considered motionless ($\vec{v}_0 = 0$) The energy of the sound $\varepsilon_2 = U_2 - \rho_0 \cdot E_0$ and the flow of sonic energy N_2 is obtained with an accuracy up to the order of magnitude A^2. Terms of the order of A^3 rejected,

$$\frac{\partial \varepsilon_2}{\partial t} + \mathrm{div}\,\vec{N}_2 = 0 \tag{1.53}$$

where

$$\varepsilon_2 = \frac{\rho_0\xi_1^2}{2} + \frac{\pi_1 + \pi_2}{\gamma - 1}$$

$$N_2 = \frac{\gamma p_0}{\gamma - 1}(\vec{\xi}_1 - \vec{\xi}_2) + \frac{\gamma\pi_1\vec{\xi}_1}{\gamma - 1} \tag{1.54}$$

Inasmuch as

$$p = p_0 + \left(\frac{dp}{d\rho}\right)_0 \cdot (\delta_1 + \delta_2) + \frac{1}{2}\left(\frac{d^2p}{d\rho^2}\right)_0 \cdot \delta_1^2 + \dots$$

$$= p_0 + c_0^2(\delta_1 + \delta_2) + \frac{1}{2}(\gamma - 1)c_0^2\delta_1^2 = p_0 + \pi_1 + \pi_2 + \dots \tag{1.55}$$

($c_0^2 = (dp/d\rho) = \gamma \cdot p_0/\rho_0$ is the square of the adiabatic velocity) and $\pi_1 = c_0^2\delta_1$, equation (1.54) may be rewritten in the form (1.54')

$$\varepsilon_2 = \frac{\rho_0\xi_1^2}{2} + \frac{\pi_1^2}{2\rho_0 c_0^2} + \frac{c_0^2}{\gamma - 1}(\delta_1 + \delta_2)$$

$$N_2 = \frac{c_0^2\rho_0}{\gamma - 1}(\vec{\xi}_1 + \vec{\xi}_2) + \frac{\gamma\pi_1\vec{\xi}_1}{\gamma - 1} \tag{1.54'}$$

For a homogeneous medium at rest ($\vec{v}_0 = 0$, c_0 = constant, and ρ_0 = constant), a new form of the conservation law follows from equation (1.53) in which the energy of the sound and its flow are expressed only in terms of the magnitudes characteristic of linear acoustics (π_1, δ_1, and $\vec{\xi}_1$), not containing the second approximations (π_2, δ_2, and $\vec{\xi}_2$). The equation of continuity expressing the law of the conservation of matter (equation (1.12)), when written with an accuracy up to terms of the order of A^2, is

$$\frac{\partial(\delta_1 + \delta_2)}{\partial t} + \rho_0 \operatorname{div} (\xi_1 + \xi_2) + \operatorname{div} (\delta_1 \xi_1) = 0 \qquad (1.56)$$

This equation is multiplied by $c_0^2/(\gamma - 1)$ and the result is subtracted from equation (1.53). Inasmuch as $\delta_1 = \pi_1/c_1^2$, equation (1.54) yields

$$\frac{\partial \varepsilon_1}{\partial t} + \operatorname{div} \vec{N}_1 = 0 \qquad (1.57)$$

where

$$\varepsilon_1 = \frac{\rho_0 \xi_1^2}{2} + \frac{\pi_1^2}{2\rho_0 c_0^2}$$

$$\vec{N}_1 = \pi \vec{\xi}_1 \qquad (1.58)$$

The new expressions obtained for the energy of sound and the energy flow ε_1 are precisely those which are applied in acoustics. In particular, if the potential φ ($\vec{\xi}_1 = -\nabla\varphi$, $\pi_1 = \rho_0(\partial\varphi/\partial t)$, see equation (1.47)) of the sound wave is introduced, then

$$\varepsilon_1 = \frac{\rho_0}{2} (\nabla\varphi)^2 + \frac{1}{2\rho_0^2 c_0^2} \left(\frac{\partial\varphi}{\partial t}\right)^2$$

$$\vec{N}_1 = -\rho_0 \frac{\partial\varphi}{\partial t} \nabla\varphi \qquad (1.59)$$

If, as is often the case, the potential φ depends harmonically on the time and is given in complex form (φ is proportional to $e^{i\omega t}$), the mean energy in time and the mean flow in time are equal to

$$\varepsilon_1 = \frac{\rho_0}{4} \nabla\varphi \cdot \nabla\varphi^* + \frac{\omega^2}{4\rho_0^2 c_0^2} \cdot \varphi\varphi^*$$

$$\vec{N}_1 = \frac{i\omega\rho_0}{4} \left\{\varphi^* \cdot \nabla\varphi - \varphi \cdot \nabla\varphi^*\right\} \qquad (1.60)$$

where the sign * indicates that the conjugate complex magnitude should be taken.

The expressions for the energy and energy flow equations, (1.54) and (1.58), are physically equivalent because the medium is supposedly homogeneous (in a nonhomogeneous medium equations (1.59) are not valid). In order to show the equivalence of the two forms of the conservation laws, one of which is a consequence of the other (under the given conditions) the radiation of sound is considered. In figure 1 is shown a sound source Q (solid body), a certain part of whose surface σ executes vibrations which excite sound waves. If the vibration started at the time instant $t = 0$, at the moment t the surface of the wave front will be the surface F (see fig. 1). The entire space between this surface and the source Q will be filled with energy radiated by the sound. With an arbitrary control surface S enclosing the sound source, the conservation theorem (1.53) is applied in integral form to the volume V included between S and Q. In order to do this, equation (1.53) must be integrated over the volume and then, the theorem of Gauss is used in transforming the integral of $\operatorname{div} \vec{N}_2$ to a surface integral. This integral will consist of the integral over the surface S and the surface of the source Q. Although some inconvenience is caused because part of this surface is movable (σ), it can easily be circumvented by the consideration that the flow of energy through the surface of the source must simply be equal to the source W_2.

From equation (1.53) the following equation is obtained:

$$\frac{\partial E_2}{\partial t} + \int_S \left[\frac{p_0 \gamma}{(\gamma - 1)} (\vec{\xi}_1 + \vec{\xi}_2) + \frac{\gamma}{(\gamma - 1)} (\pi_1 \vec{\xi}_1)_n\right] d\sigma = W_2 \qquad (1.61)$$

where n denotes the projection of $\vec{\xi}$ on the normal to the surface S, $E_2 = \int_V E_2 dv$ is the total energy of the sonic field enclosed within S; and the strength of the source Q is evidently equal to

$$W_2 = \int_\sigma \left[p_0 (\vec{\xi}_1 + \vec{\xi}_1)_\nu + (\pi \vec{\xi}_1)_\nu\right] d\sigma \qquad (1.62)$$

where ν denotes the projection on the normal to the surface σ. If the control surface is passed outside the sonic field (for example, outside the wave front F, but infinitely near it), from equation (1.61) is obtained

$$\frac{dE_2}{dt} = W_2; \quad E_2 = \int_0^t W_2 dt \qquad (1.63)$$

that is, the total radiated energy E_2 is equal to the work of the source Q. On the other hand, if the second form of the conservation law (eq. (1.18)) is treated in the same manner, the following equation results:

$$\frac{dE_1}{dt} = W_2; \quad E_1 = \int_0^t W_2 dt \tag{1.63'}$$

from which it follows that E_1 must be equal to E_2.

From equations (1.54') and (1.58),

$$E_2 - E_1 = \frac{c^2}{\gamma - 1} \int_V (\delta_1 + \delta_2) dv \tag{1.64}$$

where the integration is over the volume V. The integral $\int_V (\delta_1 + \delta_2) dv$ is the total change of mass of gas in the volume occupied by the sonic field. This change is equal to zero because the substance could not flow out beyond the limits of the wave front; hence $E_1 = E_2$. If the integral over the time period in equations (1.63) or (1.63') is taken over the entire number of periods of vibration of the source and if the fact is taken into account that in this case $d\sigma \cdot p_0 \cdot \int_0^t (\vec{\xi}_1 + \vec{\xi}_2)_\nu dt$ is equal to zero (since this integral is equal to the algebraically assumed path of a surface element $d\delta$ of the source Q in the direction along the normal to δ for a complete number of periods), and if the energy obtained over part of a period is neglected,

$$E_2 = E_1 = \int_0^t dt \int_\sigma d\sigma (\pi_1 \vec{\xi}_1)_\nu = \overline{(\pi_1 \vec{\xi})_\nu} \, \sigma t \tag{1.65}$$

where $\overline{(\pi_1 \vec{\xi})_\nu}$ is the mean value of the energy flow vector.

Both forms of the conservation law are identical when expressed in integral form. Despite the complete legitimacy and generality of the expressions for E_2 and $\vec{N}_2$ containing the elements of nonlinear acoustics, in linear acoustics it is entirely possible and more rational under the conditions of a homogeneous and stationary medium to use equations (1.58) for the energy and its flow.

The equivalence of equations (1.54) and (1.58) no longer holds if the medium is nonhomogeneous and in motion. The equations for E_2 and

$\vec{N}_2$ can easily be generalized to the case of a moving medium. Rather complicated expressions are obtained which will not be considered herein.

As will be shown in section 7, it is essential that relatively simple expressions are obtained for the energy density of sound E and energy flow $\vec{N}$ resembling expressions (1.58) and containing magnitudes of only linear acoustics in the approximation of geometrical acoustics in a nonhomogeneous and moving medium.

4. Propagation of Sound in a Nonhomogeneous Moving Medium

In the presence of air motion the acoustical phenomena become more complicated. Generally, separation of the acoustical phenomena, in the narrow sense of the word, from the doubly nonlinear processes taking place in a moving medium is not possible. Thus, for example, the flow, pulsating in velocity if the frequency of these pulsations is sufficiently large, acts on the microphone or ear located in it (not considering phenomena connected with vortex formation on the microphone body itself, see section 28) as a sound of corresponding frequency although the velocity of propagation of these pulsations has nothing in common with the velocity of sound.

The relation between the pressure of these pulsations and their velocity is nonlinear and also differs fundamentally from the relation between the pressure in a sound wave and the velocity of sound vibrations. Finally, the variable nonstationary flow itself can be a source of sound. Phenomena of this kind will be considered later but this section will be concerned exclusively with the problem of the propagation of sound. In order for it to be possible to separate the sound propagated in the medium from the acoustic phenomena arising in the same medium only as a result of its motion, this motion will be assumed to be "soundless", that is, that the motions in the flow are sufficiently slow so that

$$\tau \gg \frac{1}{f} \tag{1.66}$$

where τ is the time during which appreciable changes occur in the state of the flow (for example, the period of pulsations of the flow velocity) and f is the frequency of the sound propagated through the medium. This condition requires additional explanations. It depends on the choice of the system of coordinates to which the motion of the flow is referred.

In fact, a general translational motion of the medium has no significance since it simply leads to a transfer of the sound wave. For this reason, it is sufficient that equation (1.66) be satisfied in some one system of the uniformly moving systems of coordinates.

If, for example, a flow is considered in which the propagation of the velocities is stationary (that is, does not depend on the time, but the velocity of the flow periodically changes in space with the period l), then for this flow $\tau = \infty$. If this flow is considered from the point of view of an observer moving with velocity u, the flow will appear to him nonstationary, the period of the velocity pulsations being equal to $\tau' = l/u$.

The phenomenon of the propagation of sound in the two systems of coordinates will differ only in the transport of the sound wave as a whole with velocity u. Since for the present interest is confined to the propagation of sound, this difference, which can easily be taken into account, is not essential.

When the statement of the problem is broadened and a sound receiver is considered, entirely different results are obtained in these two reference systems. In the first system, in which the flow is stationary, the sound receiver would assume only one frequency f, the frequency of sound propagation. In the second system, in addition to this frequency[3] f the receiver would also receive the frequency of pulsations in the flow, that is, $f' = 1/\tau' = u/l$ and the combined frequencies $f_n = f \pm nf'$, $n = 1,2,3,\ldots$

In the following, condition (1.66) is assumed satisfied in any of the possible reference systems. The effect of the flow on the sound propagation will then express itself in two ways: In the first place, the sound will be "carried away" by the flow and, in the second place, it will be dissipated in the nonhomogeneities of this flow.

In the derivation of the fundamental equations of the acoustics of a moving medium, the effect of the viscosity and heat conductivity of the medium on the sound propagation is ignored. This effect, which can more conveniently be taken into account as a correction, leads to the previously considered absorption of sound. The part played by these factors, which determine irreversible processes in hydrodynamics, may be very appreciable in the formation of the initial state of the medium in which sound is propagated. No less essential in this connection is the effect of the force of gravity. Hence the theory of the propagation of sound in a nonhomogeneous and moving medium must have as its basis the general equations of motion of a compressible fluid.

According to equations (1.12), (1.13), and (1.24), these equations are

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{v}) = 0 \tag{1.67}$$

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[3]Actually it changes somewhat because of the Doppler effect; see section 5.

$$\frac{\partial \vec{v}}{\partial t} + \left[\text{rot } \vec{v}, \vec{v}\right] + \nabla \frac{v^2}{2} = - \frac{\nabla p}{\rho} + \vec{g} + \nu \Delta \vec{v} + \frac{\nu}{3} \nabla \text{ div } \vec{v} \qquad (1.68)$$

$$\frac{\partial S}{\partial t} + (\vec{v}, \nabla S) = \frac{\lambda}{\rho} \cdot \frac{\Delta T}{T} + \frac{Q}{\rho T} \qquad (1.69)$$

where $\nu = \mu/\rho$ is the kinematic viscosity of the medium. Further, equation (1.13) was supplemented by the term $+\vec{g}$, which represents the effect of the force of gravity. The vector $\vec{g}$ is the vector of the acceleration of gravity directed always toward the center of the earth. Thus $\rho \cdot \vec{g}$ is the force of gravity acting on unit volume of the fluid.

Now let sound be propagated in a medium the state of which is described by the magnitudes $\vec{v}$, p, ρ, and S. The initial state of the medium ($\vec{v}$, p, ρ, and S) is considered stable and the sound is considered as a small vibration. All the previously mentioned magnitudes will then receive small increments: $\vec{\xi}$, π, δ, and σ, respectively, where $\vec{\xi}$ will be the velocity of the sound vibrations; π, the pressure of the sound; δ, the change in density of the medium; and σ, its change of entropy occuring on passing through a sound wave.

In order to obtain the equations for the elements of the sound wave in equations (1.67), (1.68), and (1.69), $\vec{v}$ is replaced by $\vec{v} + \vec{\xi}$, p, by $p + \pi$, ρ, by $\rho + \delta$, and S, by $S + \sigma$; by restriction to a linear approximation, terms of higher order relative to the small magnitudes $\vec{\xi}$, π, δ, and σ are rejected. Moreover, as has just been mentioned, the irreversible processes taking place during the sound propagation are ignored, which means that in the linear equations for $\vec{\xi}$, π, δ, and σ the terms proportional to the viscosity (μ or ν) and the heat conductivity are rejected. On the basis of equations (1.16) and (1.5), the heat Q dissipated in the fluid likewise belongs to the number of magnitudes proportional to μ. By the method indicated,

$$\frac{\partial \vec{\xi}}{\partial t} + \left[\text{rot } \vec{v}, \vec{\xi}\right] + \left[\text{rot } \vec{\xi}, \vec{v}\right] + \nabla(\vec{v}, \vec{\xi}) = - \frac{\nabla \pi}{\rho} + \frac{\nabla p \cdot \delta}{\rho^2} \qquad (1.70)$$

$$\frac{\partial \delta}{\partial t} + (\vec{v}, \nabla \delta) + (\vec{\xi}, \nabla \rho) + \rho \cdot \text{div } \vec{\xi} + \delta \text{ div } \vec{v} = 0 \qquad (1.71)$$

$$\frac{\partial \sigma}{\partial t} + (\vec{v}, \nabla \sigma) + (\vec{\xi}, \nabla S) = 0 \qquad (1.72)$$

The equation of state, which is given in the variables ρ and S, is still to be added to these equations. For small changes of pressure π, and in exactly the same manner as in the preceding section the following is obtained:

$$\pi = c^2 \delta + h\sigma; \quad c^2 = \left(\frac{\partial p}{\partial \rho}\right)_S, \quad h = \left(\frac{\partial p}{\partial S}\right)_\rho \qquad (1.73)$$

Equations (1.70), (1.71), (1.72), and (1.73) are the fundamental equations of acoustics for a homogeneous moving medium (eq. (1.74)). Their differences from those known in the literature lie in the fact that they are true in a medium the entropy of which varies from point to point ($\nabla S \neq 0$) and in a flow in which vortices may exist (rot $\vec{v} \neq 0$).

The approximations made in these equations, in addition to linearity, consist in the fact that no account is taken of the irreversible processes in the sound wave so that the sound wave is considered an adiabatic process. This fact is also expressed by equation (1.72). In fact, it follows from this equation that $d(S + \sigma)/dt = 0$, that is, the entropy of a given amount of substance remains unchanged with the passage of a sound wave. The entropy of the substance at a given point of space may vary; $\partial\sigma/\partial t \neq 0$.

In this sense the sound wave is not isentropic. The linear character of the equations requires that a small disturbance remain small in the course of time (stability of the initial state). Hence it is not possible with the aid of these equtions to describe, for example, such interesting phenomena as the "sensitive flame" of a gas burner, the height of which changes sharply under the action of a sound wave.

In other respects the equations are entirely general and it is quite immaterial in what manner the initial state of the medium was formed. In bringing about this state, the force of gravity, the heat conductivity, and the energy flow from the outside (for example, the sun's heat) may be of considerable significance. The effect of all these factors on the sound propagation is taken into account in equations (1.70), (1.71), (1.72), and (1.73) through the magnitudes $\vec{v}$, p, ρ, and S characterizing the initial medium.

The equation $p = z(\rho, S)$ and equation (1.73) are valid only for a single-component medium. In general, the pressure may depend not only on ρ and S but also on the concentration of the various components. In a complex medium it is necessary to take into account the diffusion of the various components. The corresponding uncomplicated generalization of equations (1.70) to (1.73) will be made in section 13, where the case of sea salt water is considered.

The choice of the thermodynamic variables ρ and S that has been made herein is very convenient for general theoretical considerations. For final numerical computations, however, the variables p and T are more convenient. For this reason, formulas are given expressing the magnitudes $(\partial p/\partial S)_\rho$ and ∇S entering the equations through the variables p and T.

$$\nabla S = (\partial S/\partial T)_p \nabla T + (\partial S/\partial p)_T \nabla p$$

on the basis of the known thermodynamic relations $(\partial S/\partial T)_p = c_p/T$ (c_p is specific heat at constant pressure), $(\partial S/\partial p)_T = -(\partial V/\partial T)_p = -\beta_p/\rho$ $\left(\beta_p\right.$ is the coefficient of volume expansion and $\left.\beta_p = -\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p\right)$.

Hence

$$\nabla S = \frac{c_p}{T}\bar{\nabla} T - \frac{\beta_p}{\rho} \cdot \nabla p \tag{1.74}$$

Further,

$$(\partial p/\partial T)_S = (\partial p/\partial S)_\rho (\partial S/\partial T)_p \quad \text{and} \quad (\partial p/\partial T)_\rho = -(\partial p/\partial \rho)_T (\partial \rho/\partial T)_p$$

The magnitude

$$\left(\frac{\partial \rho}{\partial T}\right)_p = -\frac{1}{V^2}\left(\frac{\partial V}{\partial T}\right)_p = -\rho\beta_p \quad \text{and} \quad \left(\frac{\partial p}{\partial \rho}\right)_T = a^2 = \frac{c_v}{c_p}c^2$$

where c^2 is the square of the adiabatic velocity of sound and $(\partial S/\partial T)_p = c_p/T$. Thus

$$\left(\frac{\partial p}{\partial S}\right)_\rho = \frac{\rho c^2}{c_p} \cdot \beta_p T \tag{1.75}$$

On the basis of equations (1.74) and (1.75) and the medium (c^2, c_p, β_p) and its state (p and T as functions of the coordinates) ∇S and $(\partial p/\partial S)_\rho$ can easily be found.

The system of fundamental equations (1.70) to (1.73), even if, with the aid of equation (1.73), one variable is eliminated (e.q., δ), contains five unknowns and is therefore very complicated.

Nevertheless, if a complete wave picture of the propagation of sound is to be obtained, these equations cannot be avoided. The main complication lies in the fact that, because the pressure in the medium is a function of two variables (ρ and T or, preferably ρ and S), then even in a medium at rest where not only vortices of the flow are absent but where, in general, there is no flow, the right side of equation (1.70) will not be a complete differential of some function and therefore the sound will be vortical (rot $\vec{\xi} \neq 0$). Considerable simplifications are obtained when the changes in p, ρ, and S are small over the length of the sound wave. Geometrical acoustics are considered in greater detail in the next chapter.

For the present, certain special cases of the general system which are not reduced to the approximations of geometrical axoustics are considered.

The most important special case will be the one for which the initial flow is not vortical (rot $\vec{v} = 0$) and the entropy of the medium is constant ($\nabla S = 0$).

Under these conditions the pressure in the medium is a function only of the density of the medium so that $\nabla p = c^2 \nabla \rho$. From equation (1.72) it follows that for $\nabla S = 0$, $\sigma = 0$ so that the sound will be propagated isentropically. Then

$$\pi = c^2 \delta$$

If the potential of the sound pressure is introduced

$$\Pi = \frac{\pi}{\rho} \tag{1.76}$$

the right side of equation (1.70) will be equal to $-\nabla\Pi$. Therefore the velocity potential of the sound vibrations φ can also be introduced

$$\vec{\xi} = -\nabla\varphi \tag{1.77}$$

The sound will be nonvortical in this case. From equation (1.70)

$$\frac{\pi}{\rho} = \Pi = \frac{\partial\varphi}{\partial t} + (\vec{v}, \nabla\varphi) = \frac{d\varphi}{dt} \tag{1.78}$$

Substitution in equation (1.71) of the magnitude Π (for which $\partial\Pi/\partial t = (c^2/\rho) \cdot (\partial\delta/\partial t)$, $\nabla\Pi = \delta \cdot \overline{\nabla}(c^2/\rho) + (c^2/\rho) \cdot \overline{\nabla}\delta$) in place of δ yields the following equation for φ:

$$\frac{d^2\varphi}{dt^2} = c^2 \cdot \Delta\varphi + (\nabla\Pi_0, \nabla\varphi) + \frac{d\varphi}{dt} (\vec{v}, \nabla \log c^2) \tag{1.79}$$

where Π_0 is the potential of pressure (heat function) of the initial flow

$$\Pi_0 = \int \frac{dp}{\rho} \tag{1.80}$$

Equation (1.79) was derived by N. N. Andreev and I. G. Rusakov (ref. 10) without the last term, which was erroneously omitted. This equation exhaustively describes the propagation of sound in a medium in which the entropy is constant.

A. M. Obukhov (ref. 11) gives an equation which permits an approximate consideration of the presence of vorticity of the flow but nevertheless makes use of one function, the "quasipotential" ψ. This quasipotential is introduced by the equation

$$\xi = -\nabla\psi + \int^t \left[\text{rot}\ \vec{v} \times \nabla\psi\right] dt \tag{1.81}$$

The quasipotential may be introduced only for sufficiently small vorticity of the initial flow, that is, the assumption must be made that

$$\Omega = \left|\text{rot}\ \vec{v}\right| \ll \omega \tag{1.82}$$

where ω is the cyclical frequency of the sound.

Moreover the assumption is made that $v/c \ll 1$, so that the initial flow may be taken as incompressible ($\text{div}\ \vec{v} = 0$). Finally the pressure of the medium is assumed as a function of the density of the medium only. Since $\partial p/\partial \rho$ is considered by A. M. Obukhov as the adiabatic velocity of sound, this implies the assumption that the entropy of the medium is constant. In connection with this assumption, the question arises as to what extent the assumptions of the presence of vorticity ($\text{rot}\ \vec{v} \neq 0$) and the constancy of the entropy ($\overline{\nabla S} = 0$) generally apply together. The possibility is not excluded, however, that the influence of the vortices on the sound propagation is more effective than the influence of an entropy gradient. These hypotheses are assumed satisfied and $\overline{\xi}$ is substituted from equation (1.81) into equation (1.70) and, since $\nabla S = 0$, the right side of equation (1.70) will again be $= -\overline{\nabla\Pi}$. After simple reductions, the equation, which was found previously, is obtained.

$$\Pi = \frac{\pi}{\rho} = \frac{d\psi}{dt} \tag{1.83}$$

In this case, however, it is true only approximately with an accuracy to Ω^2/ω^2, $\Omega/\omega \cdot \vec{v}/c$.

Expressing δ in equation (1.71) in terms of Π and ψ gives the equation of A. M. Obukhov:

$$\frac{d^2\psi}{dt^2} = c^2\Delta\psi + (\nabla\Pi_0, \nabla\psi) + \frac{d\psi}{dt}(\vec{v}, \nabla \log c^2) +$$

$$c^2 \int^t (\nabla\psi, \Delta\vec{v}) dt - \left(\nabla\Pi_0, \int^t \left[\text{rot}\ \vec{v}, \nabla\psi\right] dt\right) \tag{1.84}$$

This equation holds with an accuracy up to Ω/ω, $\Omega/\omega \frac{\nabla|\Omega|}{k\,\Omega}$ $(k = \omega/c)$. The magnitude $\Delta\vec{v} = -$ rot rot $\vec{v}$. In this equation, the terms of order v^2/c^2 can not be taken into account because in the approximations the assumption was made that $v/c \ll 1$.[4]

5. Equation for Propagation of Sound in Constant Flow

In many cases the velocity of the flow v may be suitably separated into the mean velocity $\vec{V}$ and the fluctuating velocity u. The effect of these two components of velocity on the sound propagation may be different. The mean velocity of flow produces the "drift" of the sound wave while the second variable part of the flow velocity leads to the dissipation of the sound wave. This phenomenon will be considered in more detail later. For the present, attention is concentrated on the effect of the mean flow velocity and the equations are considered for the sound propagation, with the variable part of the flow velocity u ignored. The solution obtained under these conditions is of interest not only as a first step toward the approximate solution of the complete problem with the velocity fluctuations being considered but is of value in itself, especially for the theory of a moving sound source.

In order to obtain an equation for the propagation of sound in a homogeneous forward moving medium, it is sufficient to put $\nabla\Pi_0 = 0$ and $\nabla \log c^2 = 0$ in equation (1.79). Expansion of the total derivative with respect to time $\left[d^2\varphi/dt = \left(\partial/\partial t + (\vec{v},\nabla)(\partial\varphi/\partial t + (\vec{v},\nabla\varphi)\right)\right]$ yields

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - 2\left(\frac{\vec{v}}{c^2},\nabla\frac{\partial\varphi}{\partial t}\right) - \frac{(\vec{v},\overline{\nabla})(\vec{v}\nabla\varphi)}{c^2} = 0 \tag{1.85}$$

If the X-axis is taken in the direction of the mean velocity and β is set equal to V/c,

$$(1 - \beta^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{dz^2} - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{2\beta}{c}\frac{\partial^2\varphi}{\partial t\partial x} = 0 \tag{1.85'}$$

For the system of coordinates ξ, η, and ζ moving together with the stream $\xi = x - Vt$, $\eta = y$, and $\zeta = z$, equation (1.85') is transformed into the usual wave equation

$$\frac{\partial^2\varphi}{\partial\xi^2} + \frac{\partial^2\varphi}{\partial\eta^2} + \frac{\partial^2\varphi}{\partial\zeta^2} - \frac{1}{c^2}\cdot\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{1.86}$$

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[4]The result of A. M. Obukhov is probably more rigorous and could have successively been obtained as the second approximation of geometrical acoustics (see section 7).

as expected, since in this system of coordinates the medium is at rest. Certain important solutions of equation (1.85') are now available.

A plane sound wave is first considered. In the system of coordinates ξ, η, and ζ at rest relative to the air (hence for an observer moving with the stream), this wave has the potential

$$\varphi(\xi,\eta,\zeta,t) = Ae^{i\omega\left(t - \frac{\alpha_1\xi + \alpha_2\eta + \alpha_3\zeta}{c}\right)}; \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \tag{1.87}$$

where α_1, α_2, and α_3 are the direction cosines of the normal to the surface of the wave; ω the frequency of the oscillations; and $\underline{c}$, the velocity of sound. Equation (1.87) is a solution of equation (1.86). According to the previously mentioned transformation, the solution of equation (1.85') is immediately obtained if ξ is replaced in equation (1.87) by $x - vt$, η by y, and ζ by z

$$\varphi(x,y,z,t) = Ae^{i\left[\omega't - \frac{\omega}{c}(\alpha_1x + \alpha_2y + \alpha_3z)\right]} \tag{1.88}$$

where

$$\omega' = \omega\left(1 + \frac{V}{c}\alpha_1\right) \tag{1.89}$$

Thus the sound frequency in a stationary system of coordinates will not be ω but ω'.

This change of the frequency of the sound is the acoustical Doppler effect. The effect has an exclusively kinematic origin; it depends only on the choice of the system of coordinates. The entire difference in the propagation of a plane wave in a moving medium as compared with a stationary one reduces to this kinematic effect.

Later the Doppler effect will be considered more fully; not only the motion of the observer of the sound will be taken into account but also the motion of the sound source itself, which at present does not enter explicitly in the computation.

A second important form of the solutions of equation (1.85) is presented by sound waves diverging from a certain small point source of sound (or, on the contrary, converging to it; in the latter case a sound "sink" is being dealt with, which is a very artificial but mathematically useful concept).

The mathematical expression for the potential of such waves is a generalization of the potential of spherical waves for a medium at rest.

This potential of spherical waves is a solution of equation (1.86), having the form

$$\chi_0 = \frac{F\left(t \pm \frac{r}{c}\right)}{r}; \qquad r = \sqrt{\xi^2 + \eta^2 + \zeta^2} \tag{1.90}$$

where F is an arbitrary function. The solution with the minus sign is given by waves diverging from a sound source located at the origin of coordinates ($\xi = \eta = \zeta = 0$) and the solution with the plus sign represents the same waves converging to a sound sink at the origin of coordinates. If F is a harmonic function, the following is obtained from equation (1.90)

$$\chi_0 = \frac{e^{i\omega\left(t \pm \frac{r}{c}\right)}}{r} \tag{1.90'}$$

that is, a spherical harmonic wave with frequency ω. In a moving medium in which the propagation of sound is described by equation (1.85') instead of solutions of the form of equation (1.90), the more general expression is obtained[5].

$$\chi = \frac{F\left(t + \frac{R}{c}\right)}{R^*} \tag{1.91}$$

where

$$R = \frac{\beta x^* \pm R^*}{\sqrt{1 - \beta^2}}, \quad R^* = \sqrt{x^{*2} + y^2 + z^2}, \quad x^* = \frac{x}{\sqrt{1 - \beta^2}} \tag{1.92}$$

With the substitution of χ from equation (1.91) into equation (1.85), it is not difficult to show that equation (1.91) is in fact the solution of equation (1.85), which moreover transforms into a solution of the form of equation (1.90) for $V = 0$ ($\beta = 0$).

The solution (eq. (1.91)) for a moving medium thus has the same value which equation (1.90) has for a stationary medium; it represents waves diverging from a point source or waves converging to a sink.

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[5]The origin of this solution is clarified in detail in section 15.

6. Generalized Theorem of Kirchhoff

In the theory of the propagation of waves, an important part is played by the theorem of Kirchhoff, which permits expression of the oscillations at any point of space in terms of the oscillations at the surfaces bounding the space considered (including also the surface at infinity). This theorem is derived for a moving medium, starting from equation (1.85') (ref. 12). This equation, if the coordinate system x^*,y,z contracted in the x-direction is introduced

$$x^* = \frac{x}{\sqrt{1 - \beta^2}}; \quad y = y; \quad z = z \tag{1.93}$$

assumes the form

$$\Delta\varphi - \frac{1}{c^2} \cdot \frac{\partial^2\varphi}{\partial t^2} - \frac{2\beta}{\sqrt{1 - \beta^2}} \frac{1}{c} \frac{\partial^2\varphi}{\partial t \partial x^*} = 0 \tag{1.94}$$

where

$$\Delta = \partial^2/\partial x^{*2} + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

The singular solution χ (eq. (1.91)) likewise satisfies equation (1.94)

$$\Delta\chi - \frac{1}{c^2} \cdot \frac{\partial^2\chi}{\partial t^2} - \frac{2\beta}{\sqrt{1 - \beta^2}} \frac{1}{c} \frac{\partial^2\chi}{\partial t \partial x^*} = 0 \tag{1.95}$$

The solution χ contains the arbitrary function F which, because of later utilization of the solution for the proof of the theorem of interest, is specialized.

$$\chi = \frac{\delta\left(t + \frac{R}{c}\right)}{R^*}; \quad R = \frac{\beta x^* + R^*}{\sqrt{1 - \beta^2}} \tag{1.96}$$

where R is the distance $\sqrt{x^{*2} + y^2 + z^2}$ from the point P, with the coordinates x_P^*,y_P,z_P, at which the potential φ is to be determined to an arbitrary point of the space Q, with the coordinates x_Q^*,y_Q,z_Q, so that $x^* = x_Q - x_P^*$, $y = y_Q - y_P$, and $z = z_Q - z_P$.

The function $\delta(\xi)$ is determined such that

$$\left.\begin{aligned} &\int_a^b f(\xi)\cdot\delta(\xi)d\xi = f(0) \quad \text{if} \quad b > 0,\ a < 0 \\ &\int_a^b f(\xi)\delta(\xi)d\xi = 0 \quad \text{if} \quad \frac{b}{a} > 0 \end{aligned}\right\} \tag{1.97}$$

Equation (1.97) is assumed valid for any function $f(\xi)$ so that $\delta(\xi)$ is everywhere equal to zero except at the point $\xi = 0$, where $\delta(\xi) = \infty$. Hence $\delta(t + R/c)/R^*$ represents a converging spherical impulse (shock) concentrated about $R = -ct$.

A certain surface S enclosing the volume Ω in the space x^*,y,z is considered (see fig. 2 where the surface S is formed by two surfaces S_1 and S_2; the volume Ω is crosshatched).

After equation (1.95) is multiplied by φ and equation (1.94) by χ, one equation is subtracted from the other and the result is integrated over the volume Ω and over the time t_1 to t_2. Integration over the four-dimensional volume $\Omega(t_2 - t_1)$ yields

$$\int_{t_1}^{t_2} dt \int d\Omega(\varphi\Delta\chi - \chi\Delta\varphi) + \frac{1}{c^2}\int_{t_1}^{t_2} dt \int d\Omega\left(\chi\cdot\frac{\partial^2\varphi}{t^2} - \varphi\cdot\frac{\partial^2\chi}{\partial t^2}\right) -$$

$$\frac{2\beta}{\sqrt{1-\beta^2}}\cdot\frac{1}{c}\int_{t_1}^{t_2} dt \int d\Omega\left(\varphi\frac{\partial^2\chi}{\partial t\partial x^*} - \chi\frac{\partial^2\varphi}{\partial t\partial x^*}\right) = 0 \tag{1.98}$$

Application of Green's transformation results in

$$\int_\Omega d\Omega(\varphi\Delta\chi - \chi\Delta\varphi) = \int_S dS\left(\varphi\frac{\partial\chi}{\partial n} - \chi\frac{\partial\varphi}{\partial n}\right) \tag{1.99}$$

where $\partial/\partial n$ denotes the derivative along the external normal to the surface S enclosing the volume Ω. At the point P the transformation (eq. (1.99)) will fail because at this point χ becomes infinite. The point P is surrounded by a small surface Σ and the volume $\Delta\Omega$

enclosed by it is excluded from the volume of integration Ω in equation (1.98). The surface Σ (see fig. 2) is considered as part of the surface S. The normal to the small sphere Σ is denoted by N and directed toward the interior of the volume. If Green's transformation, equation (1.99) to equation (1.98), is applied, the following results:

$$\int_{t_1}^{t_2} dt \int_{\Sigma} dS\left(\varphi \frac{\partial \chi}{\partial N} - \chi \frac{\partial \varphi}{\partial N}\right) = \int_{t_1}^{t_2} dt \int_{S} dS\left(\varphi \frac{\partial \chi}{\partial n} - \chi \frac{\partial \varphi}{\partial n}\right) +$$

$$\frac{1}{c^2}\int_{t_1}^{t_2} dt \int_{\Omega'} d\Omega \, \frac{d}{dt}\left(\chi \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \chi}{\partial t}\right) -$$

$$\frac{2\beta}{\sqrt{1 - \beta^2}} \frac{1}{c}\int_{t_1}^{t_2} dt \int_{\Omega'} d\Omega\left(\varphi \frac{\partial^2 \chi}{\partial t \partial x^*} - \chi \frac{\partial^2 \varphi}{\partial t \partial x^*}\right) \tag{1.100}$$

The second integral on the right permits carrying out the integration with respect to time

$$I_2 = \int_{t_1}^{t_2} dt \int_{\Omega} d\Omega \, \frac{d}{dt}\left(\chi \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \chi}{\partial t}\right)$$

$$= \int_{\Omega'} d\Omega\left(\chi \frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \chi}{\partial t}\right)\Bigg|_{t_1}^{t_2} \tag{1.101}$$

But if t_1 tends to $-\infty$ and t_2 to $+\infty$ so that $t_1 + R/c < 0$ and $t_2 + R/c > 0$, then both χ and $\partial\chi/\partial t$ at t_1 and t_2 are equal to zero on account of the form chosen for χ; hence $I_2 = 0$. The first integral on the right is considered

$$I_1' = \int_{t_1}^{t_2} dt \int dS\left[\varphi \frac{\partial}{\partial n}\left(\frac{1}{R^*}\right)\delta + \varphi \frac{1}{R^*} \frac{\partial R}{\partial n} \cdot \frac{1}{c} \frac{\partial \delta}{\partial t} - \frac{\delta}{R^*} \frac{\partial \varphi}{\partial n}\right] \tag{1.102}$$

Integration by parts of the second term with respect to time and use of the property of δ (eq. (1.97)) yield

$$I_1 = \int_S dS \left\{ \frac{\partial}{\partial n}\left(\frac{1}{R^*}\right) \cdot \varphi_{t=-\frac{R}{c}} - \frac{1}{R^*}\left(\frac{\partial \varphi}{\partial n}\right)_{t=-\frac{R}{c}} - \frac{1}{c}\,\frac{1}{R^*}\,\frac{\partial R}{\partial n}\left(\frac{\partial \varphi}{\partial t}\right)_{t=-\frac{R}{c}} \right\} \tag{1.103}$$

where φ, $\partial\varphi/\partial n$, and $\partial\varphi/\partial t$ are taken at the instant $t = - R/c$.

In a similar manner the third integral on the right in equation (1.100) gives

$$\begin{aligned}
I_3 &= \frac{2\beta}{\sqrt{1 - \beta^2}} \cdot \frac{1}{c} \int_{t_1}^{t_2} dt \int d\Omega \left(\varphi \frac{\partial^2 \chi}{\partial t \partial x^*} - \chi \frac{\partial^2 \varphi}{\partial t \partial x^*}\right) \\
&= \frac{2\beta}{\sqrt{1 - \beta^2}} \frac{1}{c} \left\{ \int_{t_1}^{t_2} dt \int d\Omega \frac{\partial}{\partial x^*}\left(\varphi \frac{\partial \chi}{\partial t}\right) - \int_{t_1}^{t_2} dt \int d\Omega \frac{\partial}{\partial t}\left(\frac{\partial \varphi}{\partial x^*}\chi\right)\right\} \\
&= \frac{2\beta}{\sqrt{1 - \beta^2}} \frac{1}{c} \left\{ \int_S \varphi \frac{\partial \chi}{\partial t} dS_x - \int d\Omega \frac{\partial \varphi}{\partial x^*} \cdot \frac{\delta\left(t + \frac{R}{c}\right)}{R^*}\Bigg|_{t_1}^{t_2} \right\} \\
&= - \frac{2\beta}{\sqrt{1 - \beta^2}} \frac{1}{c} \int_S \left(\frac{\partial \varphi}{\partial t}\right)_{t=-\frac{R}{c}} \cdot \frac{1}{R^*} dS_x
\end{aligned} \tag{1.104}$$

where dS_x is the projection of the area $\vec{n}\,dS$ on the flow velocity $\vec{V}_0$ (on the x-axis). The integral in equations (1.100) on the left is transformed exactly as the first and, since in this case $\partial/\partial N$ is identical with $\partial/\partial R^*$,

$$I_0 = \int_{t_1}^{t_2} dt \int_{\Sigma} d\Sigma\left(\varphi\frac{\partial\chi}{\partial N} - \chi\,\frac{\partial\varphi}{\partial N}\right)$$

$$= \int_{\Sigma} d\Sigma\left\{\frac{\partial}{\partial R^*}\left(\frac{1}{R^*}\right)\varphi_{t=-\frac{R}{c}} - \frac{1}{R^*}\left(\frac{\partial\varphi}{\partial N}\right)_{t=-\frac{R}{c}} - \right.$$

$$\left.\frac{1}{c}\,\frac{1}{R^*}\,\frac{\partial R}{\partial R^*}\left(\frac{\partial\varphi}{\partial i}\right)_{t=-\frac{R}{c}}\right\} \tag{1.105}$$

and, since $d\Sigma = 4\pi R^{*2} \cdot dR^*$, as the radius of the sphere R^* approaches zero, the following is obtained:

$$I_0 = -\,4\pi\varphi_{t=0} \tag{1.105'}$$

Thus on the left the value of the potential at the point P at the instant of time $t = 0$ is obtained. Since this instant is arbitrary, if the time origin is everywhere shifted forward by t and all the integrals I_1, I_2, and I_3 are collected, the potential at the point P at the instant of time t will be

$$\varphi_P(t) = \frac{1}{4\pi}\int\left\{\frac{1}{R*}\left[\frac{\partial\varphi}{\partial n}\right] - \frac{\partial}{\partial n}\left(\frac{1}{R*}\right)[\varphi] + \right.$$

$$\left.\frac{1}{c}\,\frac{1}{R*}\left[\frac{\partial\varphi}{\partial t}\right]\right\}dS -$$

$$\frac{1}{4\pi}\,\frac{2\beta}{\sqrt{1 - \beta^2}}\,\frac{1}{c}\int \frac{1}{R*}\left[\frac{\partial\varphi}{\partial t}\right]dS_x \tag{1.106}$$

where the brackets indicate that the magnitude enclosed by them is taken at the instant of time $t - R/c$.

For $V_0 = 0$ $(\beta = 0)$, $R* = r$ and $R = r$ and this equation transforms into the usual equation of Kirchhoff for a medium at rest.

If the potential depends harmonically on the time so that

$$\varphi = \psi e^{i\omega t} \tag{1.107}$$

then substitution of equation (1.105) in equation (1.104) yields for the amplitude

$$\psi_P = \frac{1}{4\pi}\int\left\{\frac{\partial\psi}{\partial n}\frac{e^{-ikR}}{R^*} - \psi\frac{\partial}{\partial n}\left(\frac{e^{-ikR}}{R^*}\right)\right\}dS - \frac{2i\beta k}{4\pi\sqrt{1-\beta^2}}\int\psi\frac{e^{-ikR}}{R^*}\,dS_x \tag{1.108}$$

where $k = \omega/c$ is the wave-number vector. If, from the nature of the physical problem, it may be assumed that the disturbances giving rise to the vibrations start within the surface S_1 and not at an infinite time back, they do not have time to be propagated to the surface S_2 at a great distance from S_1. For this reason, if S_2 is shifted to infinity, the values φ, $\partial\varphi/\partial n$, $\partial\varphi/\partial t$ can be assumed equal to zero in it. The volume Ω then takes up the entire space with the exception of S_1 in the interior. If the presence of an infinitely removed surface is "forgotten," it is natural to call the normal n the interior normal since it is directed inwards from the surface S_1 within which the sources of vibration are concentrated according to the present assumption. Under this condition equations (1.104) and (1.106) may be assumed to give the expression of the potential at any point of space in terms of the values φ, $\partial\varphi/\partial n$, and $\partial\varphi/\partial t$ on the surface S_1 within which (or on it) the sound sources are concentrated.

In conclusion, a certain generalization of this theorem is considered for "volume" sources of sound. It is assumed that equation (1.94) has a right side which is considered as a "volume sound source." The strength of this source is denoted by Q. Equation (1.94) can then be written in the form

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{2\beta}{\sqrt{1-\beta^2}}\cdot\frac{1}{c}\frac{\partial^2\varphi}{\partial t\,\partial x^*} = -4\pi Q \tag{1.94}$$

Such equations are encountered, for example, in the problem of the dissipation of sound by a turbulent flow (see section 12). If the same operations which were applied to equation (1.94) are applied to this equation, an expression is obtained for φ differing from equations (1.106) and (1.108) by a volume integral. The additional term, on multiplication of equation (1.94) by χ, will be

$$I_4 = -4\pi\int_{t_1}^{t_2}dt\int d\Omega\; Q\cdot\chi \tag{1.109}$$

Integration over t yields (on account of the δ function form of χ)

$$I_4 = -4\pi \int d\Omega \, Q_{t=-\frac{R}{c}} \cdot \frac{1}{R^*} \tag{1.109'}$$

Hence, in place of equations (1.106) and (1.108), there are obtained

$$\varphi_P(t) = \int \frac{[Q]}{R^*} d\Omega + \frac{1}{4\pi} \int \left\{ \frac{1}{R^*}\left[\frac{\partial \varphi}{\partial n}\right] - \frac{\partial}{\partial n}\left(\frac{1}{R^*}\right)[\varphi] + \frac{1}{cR^*} \frac{\partial R}{\partial n}\left[\frac{\partial \varphi}{\partial t}\right]\right\} dS - \frac{1}{4\pi} \frac{-2\beta}{\sqrt{1-\beta^2}} \times \frac{1}{c} \int \frac{1}{R^*}\left[\frac{\partial \varphi}{\partial t}\right] dS_x \tag{1.106'}$$

and

$$\psi_P = \int \frac{Q_0 e^{-ikR}}{R^*} d\Omega + \frac{1}{4\pi} \int \left\{ \frac{\partial \varphi}{\partial n} \cdot \frac{e^{-ikR}}{R^*} - \varphi \frac{\partial}{\partial n}\left(\frac{e^{-ikR}}{R^*}\right)\right\} dS - \frac{1}{4\pi} \frac{2i\beta k}{\sqrt{1-\beta^2}} \int \varphi \frac{e^{-ikR}}{R^*} dS_x \tag{1.108'}$$

if the strength of the source depends harmonically on the time

$$Q = Q_0 e^{i\omega t} \tag{1.110}$$

The theorems derived herein are used in the theory of wave propagation from a moving source, in particular from an airplane propeller, and in the problem of the occurrence of vortical sound in the motion of bodies in the air.

CHAPTER II

PROPAGATION OF SOUND IN ATMOSPHERE AND IN WATER

7. Geometrical Acoustics

In the study of the propagation of sound in the atmosphere or in water, the state of the medium generally changes little over a distance equal to the length of the sound wave λ. In the background of this slow change of state of the medium there can also exist smaller changes, but these give rise to secondary effects which may be considered separately (see section 12). The main features of the sound propagation picture are determined by the slow changes in the state of the medium (for example, changes in the force of the wind and in the temperature and density of the air with increasing distance from the ground surface). Under these circumstances the application of the methods of geometrical acoustics is suitable. The fundamental equations of geometrical acoustics are derived in this section (ref. 13). A start will be made from the fundamental equations of the acoustics of a moving, nonhomogeneous medium (section 4). These equations are

$$\frac{\partial\vec{\xi}}{\partial t} + [\mathrm{rot}\ \vec{\xi}\times\vec{v}] + [\mathrm{rot}\ \vec{v}\times\vec{\xi}] + \nabla(\vec{v}, \vec{\xi}) = -\frac{\nabla\pi}{\rho} + \frac{\Delta p\delta}{\rho^2} \tag{2.1}$$

$$\frac{\partial\delta}{\partial t} + (\vec{\xi}, \nabla\rho) + (\vec{v}, \nabla\delta) + \rho\cdot\mathrm{div}\ \vec{\xi} + \delta\cdot\mathrm{div}\ \vec{v} = 0 \tag{2.2}$$

$$\frac{\partial\sigma}{\partial t} + (\vec{v}, \nabla\sigma) + (\vec{\xi}, \nabla S) = 0 \tag{2.3}$$

$$\pi = c^2\delta + h\sigma \tag{2.4}$$

The change in $\vec{v}$, p, ρ, and S is assumed small over the distance of a wavelength of sound. Use is made of this fact for the construction of an approximate theory of the propagation of sound:

$$\vec{\xi} = \vec{\xi}_0\cdot e^{i\Phi} \qquad \pi = \pi_0 e^{i\Phi} \qquad \delta = \delta_0 e^{i\Phi} \qquad \sigma = \sigma_0 e^{i\Phi} \tag{2.5}$$

$$\Phi = \omega t - k_0\Theta \tag{2.6}$$

where ω is the frequency of the sound; $k_0 = \omega/c_0 = 2\pi/\lambda_0$ is the wave number in the medium, the state of which is assumed normal (c_0 is the normal velocity of sound); and $k_0\Theta$ is the phase of the wave. The magnitudes $\vec{\xi}_0$, π_0, δ_0, and σ_0 are assumed to be slowly varying functions of the coordinates and, possibly, of the time. The number k_0 will be assumed large so that the phase $k_0\Theta$, on the contrary, varies rapidly. The solutions for $\vec{\xi}_0$, π_0, δ_0, and σ_0 will be sought in the form of series in the reciprocal powers of the large number ik_0:

$$\begin{aligned} \vec{\xi}_0 &= \vec{\xi}_0' + \frac{\vec{\xi}_0''}{ik_0} + \cdots & \pi_0 &= \pi_0' + \frac{\pi_0''}{ik_0} + \cdots \\ \delta_0 &= \delta_0' + \frac{\delta_0''}{ik_0} + \cdots & \sigma_0 &= \sigma_0' + \frac{\sigma_0''}{ik_0} + \cdots \end{aligned} \tag{2.7}$$

Substituting equations (2.5) and (2.6) in equations (2.1), (2.2), and (2.3) and making use of equation (2.4) result in

$$ik_0 \left\{q\xi_0 - \nabla\Theta \frac{\pi_0}{\rho}\right\} = \vec{v} \tag{2.8}$$

$$ik_0 \left\{\frac{q}{c^2}\pi_0 - \frac{\beta q}{c^2}\sigma_0 - \rho(\vec{\xi}_0, \nabla\Theta)\right\} = b_4 \tag{2.8'}$$

$$ik_0 q\sigma_0 = b_5 \tag{2.8''}$$

where

$$q = c_0 - (\vec{v}, \nabla\Theta) \tag{2.9}$$

$$\vec{b} = -\frac{\partial\vec{\xi}_0}{\partial t} + [\vec{\xi}_0 \times \mathrm{rot}\ \vec{v}] + [\vec{v} \times \mathrm{rot}\ \vec{\xi}_0] - \nabla(\vec{v}, \vec{\xi}_0) - \frac{\nabla\pi_0}{\rho} + \frac{\nabla p}{\rho^2} \cdot \frac{\pi_0 - h\sigma_0}{c^2} \tag{2.10}$$

$$b_4 = -\frac{1}{c^2}\frac{\partial\pi_0}{\partial t} - \frac{h}{c^2}\frac{\partial\sigma_0}{\partial t} - \left(\vec{v}, \nabla\frac{1}{c^2}\right)(\pi_0 - h\sigma_0) - \frac{1}{c^2}[\vec{v}, \nabla\pi_0 - \nabla(h, \sigma_0)] - (\nabla\rho, \vec{\xi}_0) - \rho \cdot \mathrm{div}\ \vec{\xi}_0 - \frac{\pi_0 - h\sigma_0}{c^2} \cdot \mathrm{div}\ \vec{v} \tag{2.10'}$$

$$b_5 = -\frac{\partial\sigma_0}{\partial t} - (\vec{v}, \nabla\sigma_0) - (\vec{\xi}_0, \nabla S) \tag{2.10''}$$

Substituting $\vec{\xi}_0$, π_0, and σ_0 from equations (2.7) into equations (2.8) and (2.8') and collecting the coefficients of the same powers of ik_0 give for the zeroth approximation (the coefficient of the zeroth power of ik_0)

$$q\vec{\xi}_0' - \nabla\Theta \cdot \frac{\pi_0'}{\rho} = 0 \tag{2.11}$$

$$q/c^2(\pi_0' - h\sigma_0') - \rho(\vec{\xi}_0', \nabla\Theta) = 0 \tag{2.11'}$$

$$q\sigma_0' = 0 \tag{2.11"}$$

and for the first approximation (the coefficient of the first power of ik_0)

$$q\vec{\xi}_0'' - \nabla\Theta \cdot \frac{\pi_0''}{\rho} = \vec{b}' \tag{2.12}$$

$$q/c^2(\pi_0'' - h\sigma_0'') - \rho(\xi_0'', \nabla\Theta) = b_4' \tag{2.12'}$$

$$q \cdot \sigma_0'' = b_5' \tag{2.12"}$$

where $\vec{b}'$, b_4', and b_5' are the values of $\vec{b}$, b_4, and b_5 on substituting in them the zeroth approximation of $\vec{\xi}_0'$, π_0', and σ_0' from equations (2.11), (2.11'), and (2.11").

From equation (2.11"), it follows that $\sigma_0' = 0$, that is, <u>in the zeroth approximation of geometrical acoustics the sound is propagated without change of entropy (isentropically).</u>

Solving equations (2.11), (2.11'), and (2.11") gives, in the first place, the equations connecting the velocity of the oscillations with the pressure

$$\vec{\xi}_0' = \nabla\Theta \cdot \frac{\pi_0'}{\rho_1} \tag{2.13}$$

and as the condition of the simultaneity of equations (2.11) and (2.11'), the equation of the surface of constant phase (Θ = constant) is

$$\left|\nabla\Theta\right|^2 = \frac{q^2}{c^2} \tag{2.14}$$

For $v = 0$, as is seen from equation (2.9), $q^2/c^2 = c_0^2/c^2 = \mu^2$, where μ is the refraction index for sound waves. The equation

$|\nabla\Theta|^2 = \mu^2$ is called the "eikonal" equation. For $v \neq 0$, the ratio q/c may likewise be considered the index of refraction of the medium, but it now depends also on the direction of propagation of the waves.

The situation is similar to that in crystal optics, but more complicated because for acoustics the medium is not only anisotropic but also nonhomogeneous, since the position of the axis coincides with the position of the wind or flow which changes from point to point. Substituting in equation (2.14) the value of q from equation (2.9) and solving equation (2.14) for $|\nabla\Theta| = \partial\Theta/\partial n$, where $\partial\Theta/\partial n$ denotes differentiation along the direction of the normal to the surface of constant phase (Θ = constant), give

$$|\nabla\Theta| = \frac{\partial}{\partial n} = \frac{c_0}{c + v_n} \tag{2.15}$$

where v_n is the projection of the velocity of the wind on the normal to the wave. With $\partial\Theta/\partial n$ known, the phase velocity of the waves V_f can be determined. The equation of the moving phase surface is $\Phi = \omega t = k_0\Theta$ = constant. Differentiating this equation with respect to time results in

$$\omega - k_0 \frac{\partial\Theta}{\partial n} \cdot \frac{dn}{dt} = \omega - k_0 \frac{\partial\Theta}{\partial n} V_f = 0 \tag{2.16}$$

On the basis of equation (2.15) there is then obtained

$$V_f = c + v_n \tag{2.17}$$

that is, the phase velocity of the waves is equal to the local velocity of the sound plus the projection of the velocity of the wind on the normal to the wave. This kinematic relation is clarified in figure 3; equation (2.17), which was obtained as a consequence of the strict theory, was put at the basis of a geometrical theory of sound propagation as one of the initial assumptions by R. Emden (ref. 14).

It is important, however, not only to find the geometry of the wave field but also to compute the magnitudes characterizing the intensity of the sound. The equation for the determination of the sound pressure π_0 is obtained from the equations of geometrical acoustics (2.11) and (2.12). This magnitude is generally measured in a test. The equations of the second approximation (2.12) are used to obtain this equation. The left sides of these equations agree with equations (2.11). If the notations $\vec{\xi}_0' = (x_1', x_2', x_3')$, $\pi_0' = x_4'$, and $\sigma_0' = x_5'$ are introduced and equations (2.11) are written in the form

$$\sum_{k=1}^{5} a_{ik} \cdot x_k' = 0 \qquad i = 1, 2, 3, 4, 5 \tag{2.18}$$

equations (2.12) can be written in the form

$$\sum_{k=1}^{5} a_{ik} \cdot x_k'' = b_i' \qquad i = 1, 2, 3, 4, 5 \tag{2.18}$$

By a known theorem of algebra, equations (2.18) will have solutions x_k'' only when the right sides are orthogonal to the solutions y_k of the adjoint system of equations:

$$\sum_{k=1}^{5} \tilde{a}_{ik} \cdot y_k = 0 \quad \text{where} \quad \tilde{a}_{ik} = a_{ki} \tag{2.19}$$

The condition of orthogonality is

$$\sum_{k=1}^{5} b_k' y_k = 0 \tag{2.20}$$

With a_{ik} determined from equations (2.11), (2.11'), and (2.11") and a_{ik} transformed, y_k is obtained from equations (2.19) in the form

$$\vec{y} = \rho \cdot \nabla\Theta, \qquad y_4 = q, \qquad y_5 = \frac{h}{c^2} q \tag{2.21}$$

Substituting $\vec{b}$, b_4, and b_5 from equation (2.10) in equation (2.20) and making use of equation (2.13) give the condition of orthogonality (eq. (2.20)) in expanded form:

$$2 \frac{\partial\pi_0'}{\partial t} + 2\pi_0' \operatorname{div} \vec{V}_S + 2\vec{V}\nabla\pi_0' - (\vec{V}_S, \nabla \log \rho qc^2) \cdot \pi_0' = 0 \tag{2.22}$$

where the velocity $\vec{V}_S$ is given by (see fig. 3)

$$\vec{V}_S = c\vec{n} + \vec{v} \tag{2.23}$$

$\vec{n}$ being the unit vector along the normal to the surface of constant phase.

Dropping the strokes of π_0' and $\vec{\xi}_0'$, because the zeroth approximation is concerned in what follows, equation (2.22) is multiplied by π and an equation for the square of the pressure amplitude is obtained:

$$\frac{\partial \pi^2}{\partial t} + \operatorname{div}\,(\vec{V}_S \pi^2) = (\vec{V}_S,\ \nabla \log \rho q c^2)\pi^2 \tag{2.24}$$

which together with equation (2.13)

$$\vec{\xi} = \nabla\Theta \frac{\pi}{\rho q} \tag{2.25}$$

completely solves the problem of obtaining the sound pressure π and the velocity of the sound vibrations $\vec{\xi}$. Equation (2.24) may be considered also as a certain conservation law. In fact, the mean kinetic energy of the sound vibrations T is defined by the equation

$$T = \frac{1}{2}\,\overline{(\rho + \delta)(\vec{v} + \vec{\xi})^2} - \frac{\rho v^2}{2} = \frac{1}{2}\,\rho\overline{\vec{\xi}^2} + \overline{\delta(\vec{v},\ \vec{\xi})} \tag{2.26}$$

where the remaining terms are rejected either as magnitudes of third-order smallness or as magnitudes which within the framework of the linear theory should, on the average, give zero (for example, $\overline{\rho(\vec{v},\ \vec{\xi})}$). Since $\delta = \pi/c^2$ (compare eq. (2.4)),

$$T = \frac{1}{2}\left|\nabla\Theta\right|^2 \cdot \frac{\pi^2}{\rho q^2} + (\vec{v},\ \nabla\Theta) \cdot \frac{\pi^2}{\rho q c^2} \tag{2.27}$$

Adding the mean potential energy of the second order U

$$U = \frac{1}{2}\,\frac{\pi^2}{\rho c^2} \tag{2.28}$$

results, on the basis of equations (2.9) and (2.14), in

$$\varepsilon = T + U = \frac{\pi^2 \cdot c_0}{\rho q c^2} \tag{2.29}$$

If equation (2.24) is divided by $\rho q c^2/c_0$, then after simple reductions,

$$\frac{\partial \varepsilon}{\partial t} + \operatorname{div}(\varepsilon \vec{V}_S) = 0 \tag{2.30}$$

that is, the law of conservation of the average energy in geometrical acoustics. This law, like the law for ε_1 and $\vec{N}_1$ (see section 3), is remarkable in that it contains only magnitudes characteristic for linear acoustics. It is valid for any nonhomogeneous and moving medium provided only that the length of the sound wave is sufficiently small that the approximations of geometrical acoustics are applicable.

The magnitude $\varepsilon\vec{V}_S$ is evidently the mean energy flow

$$\vec{N} = \varepsilon\vec{V}_S \tag{2.31}$$

It follows immediately that the sound energy is propagated with the velocity $\vec{V}_S = c\vec{n} + \vec{v}$, different from the phase velocity $\vec{V}_f$. The velocity $\vec{V}_S$ is called the ray velocity. This velocity is equal to the geometric sum of the local sound velocity $c\vec{n}$ and the wind velocity $\vec{v}$. It coincides with the velocity of weak explosions according to Hadamard (ref. 15).

On the basis of equations (2.23) and (2.25), the energy flow may also be represented in the form

$$\vec{N} = \left(\pi\vec{\xi} + \frac{\pi^2}{\rho c^2}\vec{v}\right) \cdot \frac{c_0}{q} \tag{2.31'}$$

For $v = 0$, $q = c_0$; and the previously derived (section 3) equation for the flow $\vec{N} = \pi\vec{\xi}$ is obtained (the expression $\vec{N}_1 = \pi_1\vec{\xi}_1$ differs, however, from $\vec{N} = \pi\vec{\xi}$ since the latter vector represents the average value in time of the energy flow while $\vec{N}_1$ is its instantaneous value). If the process is stationary, so that the mean energy of the sound field does not change (at least where the sound field has already filled the space), from equation (2.30),

$$\operatorname{div}\,(\varepsilon\vec{V}_S) = 0 \tag{2.30'}$$

From this equation it follows that, if tubes are constructed the lateral surfaces of which are formed by lines along which the ray velocity is directed ("ray tubes," fig. 4), the product $\varepsilon \cdot V_S, s$(s is the cross section of the tube) is constant

$$\varepsilon V_S s = \text{constant} \tag{2.32}$$

Substituting the value of ε from equation (2.29) gives

$$\pi^2 V_S s = \pi_1^2 V_{s1} s_1 \cdot \frac{\rho q c^2}{\rho_1 q_1 c_1^2} \tag{2.33}$$

where π_1, V_{s1}, s_1, ρ_1, q_1, and c_1 are values of these magnitudes at any chosen section of the tube. This equation permits computation of the pressure of the sound at any part of the ray tube as soon as it is known at any section of it. To obtain the geometry of the ray tubes, however, a solution of the problem of geometrical acoustics (equation of the eikonal (2.14)) is required.

8. Simplest Cases of Propagation of Sound

A. Propagation in an isothermal atmosphere. - In an isothermal atmosphere at rest, the velocity of sound is constant (since it depends only on the temperature). Thus $c = c_0 = \text{constant}$. The magnitude $q = c_0$ (since $\vec{v} = 0$). Hence, from equation (2.33) for the conditions considered,

$$\pi^2 s = \pi_1^2 s_1 \cdot \rho/\rho_1 \tag{2.34}$$

In the special case of a plane wave, the cross section of a tube is constant ($s = s_1$) and

$$\pi = \pi_1 \cdot (\rho/\rho_1)^{1/2} \tag{2.35}$$

that is, the pressure of the sound is directly proportional to the square root of the density of the medium. The ratio ρ/ρ_1 in an isothermal atmosphere is determined by the barometric formula

$$\rho/\rho_1 = e^{-\varkappa \cdot H} \tag{2.36}$$

where $\varkappa = M \cdot g/RT$, H is the altitude, M is the molecular weight of the air, g is the acceleration of the force of gravity, R is the constant gram molecular weight of the gas, and T is the temperature. From equations (2.35) and (2.36) it is seen that the pressure will decrease with altitude by the exponential law.

If the wave is not plane but spherical, the cross section of the tubes increases as the square of the distance from the source r^2. Hence for a spherical wave in place of equation (2.35),

$$\pi = \pi_1 \cdot \frac{r_1}{r} (\rho/\rho_1)^{1/2} \tag{2.35'}$$

The velocity of the sound vibrations $\vec{\xi}$, in contrast to the pressure, will increase. In fact, for a plane wave $\vec{\nabla}\Theta = \vec{n}$ ($\vec{n}$ is the unit vector in the direction of the normal to the wave) and therefore from equations (2.25) and (2.35) there follows

$$\vec{\xi} = \vec{n} \frac{\pi_1}{\rho c_1} (\rho/\rho_1)^{1/2} = \vec{n} \cdot \frac{\pi_1}{\rho_1 c_1} \cdot (\rho_1/\rho)^{1/2} \tag{2.37}$$

The mean energy flow

$$\vec{N} = \pi\vec{\xi} = \vec{n} \cdot \frac{\pi_1^2}{\rho_1 c_1} \tag{2.38}$$

remains constant.

In a similar manner, for the spherical wave,

$$\vec{\xi} = \vec{n}\,\frac{\pi_1}{\rho_1 c_1}\cdot\frac{r_1}{r}\,(\rho_1/\rho)^{1/2} \tag{2.37'}$$

$$\vec{N} = \pi\vec{\xi} = \vec{n}\,\frac{r_1^2}{r^2}\,\frac{\pi_1^2}{\rho_1 c_1} \tag{2.38'}$$

where $\vec{n}$ is again the unit vector along the normal to the wave, that is, in the direction of a ray issuing from the source.

B. Case of the presence of a temperature gradient. - Let the temperature T be a function of the altitude y. The velocity of the sound c will then vary according to the law

$$c = \sqrt{\gamma\cdot\frac{p}{\rho}} = \sqrt{\gamma r T} \tag{2.39}$$

and the index of refraction of the sound wave μ will be

$$\mu = \frac{c_0}{c} = \sqrt{\frac{T_0}{T}} \tag{2.40}$$

The equation of the surface of constant phase (equation of the eikonal) in the absence of wind will, according to equation (2.14), read

$$\left(\frac{\partial\Theta}{\partial x}\right)^2 + \left(\frac{\partial\Theta}{\partial y}\right)^2 = \mu^2 = \frac{T_0}{T} \tag{2.41}$$

(The x-axis is directed horizontally (fig. 5) in the plane of the sound ray and therefore it is assumed that Θ does not depend on z.) The cosine of the angle φ between the x-axis and the normal to the wave will be

$$\cos\varphi = \frac{\partial\Theta}{\partial x}\Bigg/\sqrt{\left(\frac{\partial\Theta}{\partial x}\right)^2 + \left(\frac{\partial\Theta}{\partial y}\right)^2} \tag{2.42}$$

Let $\partial\Theta/\partial x = \cos\varphi_0$, where φ_0 is the value of φ for $y = 0$, that is, on the ground surface, where $T = T_0$. From equations (2.41) and (2.42),

$$\cos \varphi = \cos \varphi_0 \cdot \sqrt{\frac{T}{T_0}} \tag{2.43}$$

From this equation it is seen that, if, as is generally the case, the temperature drops with the altitude, $\cos \varphi$ will decrease in absolute magnitude and therefore the ray will be deflected from its initial direction upward (fig. 5). By use of equation (2.43), if the temperature distribution over the layers is known, the entire curve of the ray can be constructed.

C. Propagation of sound for a stratified wind. - The case of a medium of constant temperature and density wherein there is a horizontal wind (let it be directed along the x-axis) the force of which varies with the altitude is now considered.

Let the velocity of the wind be

$$v = v(y) \tag{2.44}$$

Then according to equation (2.49), the magnitude q is equal to

$$q = c_0 - v(y) \frac{\partial \Theta}{\partial x} \tag{2.45}$$

and on the basis of equation (2.14), the equation of the eikonal will be

$$\left(\frac{\partial \Theta}{\partial x}\right)^2 + \left(\frac{\partial \Theta}{\partial y}\right)^2 = \frac{q^2}{c^2} = \left[1 - \gamma(y) \frac{\partial \Theta}{\partial x}\right]^2 \tag{2.46}$$

where $\gamma(y) = v(y)/c_0$.

The velocity of the wind at the ground surface itself $(y = 0)$ will be assumed equal to zero $(\gamma(0) = 0)$. Assuming, also, as in (B), that the initial angle of the normal to the wave is equal to φ_0, $\partial\Theta/\partial x$ is set equal to $\cos \varphi_0$ and from equation (2.46) is obtained

$$\cos \varphi = \frac{\cos \varphi_0}{\left|1 - \cos \varphi_0 \gamma\right|} \tag{2.47}$$

From this equation it follows that if the ray[6] is directed along the wind $(\gamma \cdot \cos \varphi_0 > 0)$, then as the velocity of the wind increases with

[6] In the presence of a wind, as was already pointed out, the line of the ray differs from the line of the normal. Since, however, $v/c \ll 1$, this difference is not large.

the altitude, $\cos \varphi$ increases in such a manner that the ray is deflected toward the earth (fig. 6), while a ray traveling against the wind is deflected upward. This upward deflection is one of the reasons for the impairment of heating in a wind. Consider a ray which in the absence of wind almost glides over the surface of the earth (fig. 7).

In the presence of a wind the force of which increases with the altitude, this ray is deflected upward and passes by the receiver P. This does not mean, of course, that at P nothing will be heard since other rays will arrive there, but the intensity of the sound will be considerably weakened (small number of rays). If the force of the wind drops with the altitude, the same conclusion will hold for the propagation of the sound along the wind direction.

In those cases where not only the force of the wind but also its direction varies from layer to layer, the picture of the sound propagation becomes considerably more complicated because the rays will be curves of double curvature.

9. Propagation of Sound in a Real Atmosphere. Zones of Silence

Under the conditions of the real atmosphere all the factors considered (wind, temperature gradient) act simultaneously and in a very complicated manner since the variation of the temperature, force, and direction of the wind may be very different. In the general case the direction cosines of the normal to the wave α, β, and γ are again determined from equation (2.14). Since $c_0/c = \sqrt{T_0/T}$ and $q = c_0 - (\nabla\Theta, \vec{v})$,

$$\left.\begin{aligned}
\alpha &= \frac{\partial\Theta}{\partial x} \sqrt{\frac{T}{T_0}} \frac{1}{\left(1 - \frac{\nabla\Theta\vec{v}}{c}\right)} \\
\beta &= \frac{\partial\Theta}{\partial y} \cdot \sqrt{\frac{T_0}{T}} \frac{1}{\left(1 - \frac{\nabla\Theta\vec{v}}{c_0}\right)} \\
\gamma &= \frac{\partial\Theta}{\partial z} \sqrt{\frac{T_0}{T}} \frac{1}{\left(1 - \frac{\nabla\Theta\vec{v}}{c_0}\right)}
\end{aligned}\right\} \qquad (2.48)$$

For their determination, it is thus necessary to know the function Θ from equation (2.14).

As could have been seen from the equations of the preceding section, an essential part in the propagation of sound is played not so much by the temperature and the force of the wind as by their change. It is found that negligible gradients of the temperature or of the wind force lead to considerable curvature of the sound rays.

Several illustrations borrowed from the paper by R. Emden (ref. 14) are presented. In figure 8 is represented the case of the propagation of sound in an atmosphere in which the temperature drops by 6.2° in 1 kilometer; on the ground surface up to an altitude of 370 meters there is assumed a calm, but further on the velocity of the wind increases by 4 meters per second per kilometer. In this case there is formed a wide "zone of silence" lying to the right of the sound source. The sound reaches the surface of the ground only at a considerable distance from the sound source (beyond 159 kilometers). Similar regions of sound shadows are seen in figure 9 where sound rays are shown propagated in an atmosphere in which up to a height of 910 meters the temperature drops by 3° while the wind increases by 2.13 meters per second, and higher up the temperature drops by 3.65° in 1 kilometer and the wind velocity likewise drops by 3.28 meters per second in 1 kilometer. Zones of silence were first observed in the last war when it was found that the audibility of an artillery cannonade was greater at places further removed from the sound source than in its neighborhood.

Very brilliant and detailed computations of the propagation of a sound-wave front in a nonhomogeneous atmosphere in the presence of wind may be found by the reader in the work of S. V. Chibisov (ref. 16) in which examples of zones of silence are likewise given.

The velocity of propagation of weak explosions (according to Hadamard) which figures in the work of Chibisov agrees (ref. 12) with the ray velocity V_S introduced in section 7. Since it is not possible to enter into more detail in regard to the computational problems of air seismics, the discussion of these problems is limited to the illustrations given and to the references cited.

10. Turbulence of the Atmosphere

The propagation of sound in a medium the state of which changes little over the distance of a sound-wave length was considered in the preceding section. In the real atmosphere such a method of treatment gives only the main features of the sound propagation. As a matter of fact, in addition to the slow change of state of the atmosphere from one layer to the next, there are also more rapid changes brought about by accidental fluctuations in the velocity of the wind, namely, the turbulence of the atmosphere. These changes may be very rapid and their effect on the sound propagation can by no means always be considered by the methods of geometrical

acoustics since the dimensions of the region in which an appreciable change of state of the medium occurs may be entirely comparable with the length of the sound wave.

Before considering the effect of these phenomena on the sound propagation, the fundamental laws of turbulence are considered. The theory of turbulence forms a very extensive and as yet far from fully developed field of hydrodynamics and aerodynamics. At the end of this chapter the reader will find references to the fundamental literature on this subject.

The work of A. N. Kolmogorov, M. D. Millionshchikov, and A. M. Obukhov in recent times has greatly contributed to the development of the theory of turbulence. The scope and purpose of this book do not permit any detailed consideration of these works.

The discussion is restricted to what is most required for present purposes without pretense of mathematical rigor.

The velocity in a turbulent flow $\vec{v}(\vec{x})$ is a random function. The entire velocity field of such a flow may be represented as a system of disturbances ("vortices") of different scales. The largest vortices are defined by the dimensions of the entire flow as a whole L. The meaning of the magnitude L may be very different. For example, it may be the height of a layer of air above the surface of the ground, the dimensions of the body, or, it the turbulence is brought about from the initially laminar flow about the body, the dimensions of the pipe from which the stream issues, and so forth.

These large-scale disturbances break up into smaller vortices and the dimensions of the smallest are determined by the viscosity of the medium, since very sharp changes in the motion of the medium rapidly die down precisely on account of the viscosity (compare with the dissipative function Q introduced in section 1 from which it is seen that the energy of the flow converted into heat because of the action of the viscosity is greater the greater the gradient of the flow velocity).

Such a picture of the distribution of the velocities of a turbulent flow over different scales of disturbances with successive conversion of the energy of the large disturbances into the energy of small disturbances and finally into heat was first clearly described by Richardson.

In order to characterize mathematically the spectral distribution of the velocity of the turbulent flow $\vec{v}(\vec{x})$ over the different scale disturbances, the velocity $\vec{v}(\vec{x})$ is expressed as a Fourier type integral

$$v_i(\vec{x}) = \int e^{i(\vec{q},\ \vec{x})} \cdot U_i(d\Omega(\vec{q})) \tag{2.49}$$

where $v_i(\vec{x})$ denotes a component of the velocity of the turbulent flow ($i = 1,2,3$ are the numbers of the axes ox, oy, oz), $\vec{q}(q_1, q_2, q_3)$ is the wave vector belonging to the scale $l = 2\pi/q$, and $d\Omega(\vec{q})$ is an element of volume in space of wave number $\vec{q}$. Finally, $U_i(d\Omega(\vec{q}))$ is the (infinitely small) Fourier amplitude defining the magnitude of the velocity pulsations of scale l. It is an additive function of the volume $d\Omega$:

$$U_i(\Omega_1 + \Omega_2) = U_i(\Omega_1) + U_i(\Omega_2) \tag{2.49'}$$

If $v_i(x)$ were a continuous function of the point x, there could be written: $U_i(d\Omega(\vec{q})) = \vec{v}_i(\vec{q}) \cdot d\Omega(V_i(\vec{q}))$; the "density" of the velocity in space $\vec{q}$ and the additive property would then be trivial since

$$U_i(\Omega_1 + \Omega_2) = \int_{\Omega_1+\Omega_2} v_i d\Omega$$

$$= \int_{\Omega_1} v_i d\Omega + \int_{\Omega_2} v_i d\Omega = U_i(\Omega_1) + U_i(\Omega_2) \tag{2.49''}$$

The density v_i, however, may not exist while the additive property, as a more general one, may be maintained (for example, discontinuous functions).

In particular in this case, $U_i(d\Omega)$ is a random function (in the space $\vec{q}$) and cannot, in general, be assumed as continuous. Hence it is necessary to make use not of the Fourier integral but of the more general expression (2.49)[7].

The following assumptions are made relative to the statistical properties of U_i:

(1) The velocity fluctuations associated with the different scales are statistically independent so that the mean of $U_i(\Omega_1) \cdot U_k^*(\Omega_2)$ is equal to zero

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[7]With regard to the mathematical basis of the expression of a random function as an integral (2.49), see A. N. Kolmogorov (ref. 18). In the following discussion, the presentation of A. M. Obukhov (ref. 19) is followed (essentially). The same results, but by a somewhat different method, were obtained also by Kolmogorov (ref. 20).

$$\overline{U_i(\Omega_1) \cdot U_k^*(\Omega_2)} = 0 \tag{2.50}$$

if the volumes Ω_1 and Ω_2 do not overlap (which means that the U_i and U_k belong to different q). The asterisk * denotes the conjugate complex magnitude.

(2) For coinciding volumes it is assumed that

$$\overline{U_i(\Omega_1)U_k^*(\Omega_1)} = \Phi_{ik}(\Omega_1) \tag{2.51}$$

is an additive function of the region Ω. Physically this means that the intensities associated with the different scales of turbulence are combined. Since Φ_{ik} is a certain mean magnitude, it may be a smooth function and may be expressed in terms of the "density" ψ_{ik}:

$$\Phi_{ik}(\Omega) = \int_\Omega \psi_{ik}(\vec{q})d\Omega \tag{2.52}$$

The value ψ_{ik} shall be called the spectral tensor since it determines, as will be seen, the distribution of the energy in a turbulent flow over the different scales of the fluctuations $l = 2\pi/q$. If interest lies not in the complete velocity of the turbulent flow but in only that part of it $\vec{v}^p(\vec{x})$ which refers to the velocity fluctuations having a scale less than $l = 2\pi/p$, the expression for $\vec{v}^p(\vec{x})$ is obtained from equations (2.48) if the integration with respect to q is extended over the range $q > p$:

$$v_i^p(\vec{x}) = \int_{q>p} e^{i(\vec{q},\ \vec{x})}U_i(d\Omega(\vec{q})) \tag{2.53}$$

The "moments of correlation" $M_{ik}^p(\vec{x}', \vec{x}'')$ are determined by the equation

$$M_{ik}^p(\vec{x}', \vec{x}'') = \overline{v_i^p(\vec{x}') \cdot v_k^p(\vec{x}'')} \tag{2.54}$$

that is, as the mean of the product of two velocity components v_i^p and v_k^p taken at two different points $\vec{x}'$ and $\vec{x}''$. The set of magnitudes $M_{ik}(\vec{x}', \vec{x}'')(i, k = 1, 2, 3)$ forms the tensor of the correlation moments. For homogeneous turbulence, that is, such that the states of the flow at different points of space do not differ from one another, the tensor of the correlation moments will depend only on the distance between the

points $\vec{x}'$ and $\vec{x}''$, that is, on $\vec{\rho} = \vec{x}' - \vec{x}''$. Substituting $v_i^p(\vec{x})$ from equation (2.53) into equation (2.54) gives

$$M_{ik}^p(\vec{\rho}) = \int_{q \geq p}^{\vec{\rho}} e^{i(\vec{q},\, \vec{x}+\vec{\rho})} U_i(d\Omega(q')) \cdot \int_{q \geq p} e^{-i(\vec{q},\, \vec{x})} \cdot U_k^*(d\Omega(\vec{q}'')) \tag{2.55}$$

Use is made of the statistical independence of U_i and U_k belonging to different $\vec{q}$ (condition (2.50)) and of the additivity (conditions (2.49), (2.51), and (2.52)) to obtain

$$M_{ik}^p(\vec{\rho}) = \int_{q>p} e^{i(\vec{q},\, \vec{\rho})}\, \psi_{ik}(\vec{q})\, d\Omega \tag{2.56}$$

The motion of the fluid is considered incompressible so that $\operatorname{div} \vec{V} = 0$. From equation (2.53) there then follows:

$$\sum_{i=1}^{3} \frac{\partial \vec{v}_i^p(\vec{x}')}{\partial x_i'} = 0 \tag{2.57}$$

Applying this relation twice to equation (2.54) (differentiating once with respect to $\vec{x}'$ and again with respect to $\vec{x}''$) results in

$$\sum_{i,\, k=i}^{3} \frac{\partial^2 M_{ik}^p(\vec{x}', \vec{x}'')}{\partial x_i' \cdot \partial x_k'} = 0 \tag{2.58}$$

From the preceding and from equation (2.56) it then follows that the spectral tensor $\psi_{ik}(\vec{q})$ must have the form

$$\psi_{ik}(\vec{q}) = \left(\delta_{ik} - \frac{q_i q_k}{q^2}\right) f(q) \tag{2.59}$$

This tensor is now connected with the energy distribution in a turbulent flow over the fluctuations of different scales l. The energy shall be considered as referred to unit mass so that the measure of energy will be $v^2/2$. The mean energy $E(p)$, referring to the velocity fluctuations the dimensions of which are less than $l = 2\pi/p$, will be equal to

$$E(p) = \frac{1}{2}\overline{(v^p(\vec{x}))^2} = \frac{1}{2}\sum_{i=1}^{3}\overline{(v_i^p(x))^2} = \frac{1}{2}\sum_{i=1}^{3} M_{ii}^p(0) = \frac{1}{2}\sum_{i=1}^{3}\int_{q>p}\psi_{ii}(\vec{q})d\Omega \tag{2.60}$$

or on the basis of equation (2.59),

$$E(p) = 4\pi\int_{q>p}^{\infty} f(q)q^2dq \tag{2.61}$$

For determining the form of $E(p)$, use is made of dimensional considerations. The flow is assumed not only homogeneous but also isotropic (of course again statistically in the mean). The turbulent motion of such a flow must be maintained by a certain constant supply of energy from outside, for example, by the energy of solar radiation giving rise to the motion of air currents.

This same energy, since a stationary state is considered, is dissipated in turbulent motion, being converted because of the action of viscous stresses into heat. The energy dissipated shall be denoted in unit time (per unit mass of gas) by D_0. (It is equal to the supply of energy from outside.) The dimensions of D_0 are $-L^2T^{-3}(cm^2/sec^3)$. In a developed homogeneous and isotropic turbulence its spectral state must be determined by the supply of energy which maintains the turbulence, that is, $E(p) = F(D_0,p)$. Representing F in the form $D_0^n \cdot p^m$, a dimensional equation for determining n and m is obtained in the form

$$L^2T^{-2} = (L^2T^{-3})^nL^m \tag{2.62}$$

from which $n = 2/3$ and $m = -2/3$. The impossibility of forming any nondimensional combinations from D_0 and p leaves

$$E(p) = \text{constant } D_0^{2/3}\ p^{-2/3} \tag{2.63}$$

A more detailed analysis by A. M. Obukhov (loc. cit.) shows that constant = $\sqrt[3]{2} \cdot \varkappa^{-2/3}$ where $\varkappa$ is a certain nondimensional number of the order of 1; thus, in the notation of Obukhov,

$$E(p) = \sqrt[3]{2} \cdot \left(\frac{D_0}{\varkappa}\right)^{2/3} p^{-2/3} \tag{2.63'}$$

Since $p = 2\pi/l$, $E(P) \cong l^{2/3}$.

This law, established by A. M. Obukhov (ref. 19) and A. N. Kolmogorov (ref. 20) is usually briefly referred to as the "2/3" law. From the law it follows that the energy of homogeneous and isotropic turbulence is concentrated mainly in the region of large-scale fluctuations of the velocity. The value of the energy $E(l)$ is restricted by the maximum scale of the turbulence L determining the dimension of the flow as a whole. For atmospheric turbulence L is the height of observation above the earth's surface.

Differentiating equation (2.61) with respect to p and using equation (2.63) give

$$f(p) = \gamma p^{-11/3} \qquad \gamma = \frac{2\sqrt[3]{2}}{3}\left(\frac{D_0}{\varkappa}\right)^{2/3} \tag{2.64}$$

and therefore the spectral tensor is equal to

$$\Psi_{ik}(\vec{q}) = \left(\delta_{ik} - \frac{q_i q_k}{q^2}\right)\gamma\vec{q}^{\,-11/3} \tag{2.65}$$

In concluding, the mean-square difference of the velocity component taken at two different points of space is computed:

$$\overline{\left(v_i^p(\vec{x}') - v_i^p(\vec{x}'')\right)^2} = \overline{2\left(\left(v_i^p(\vec{x}')\right)^2 - v_i^p(\vec{x}')v_i^p(\vec{x}'')\right)} \tag{2.66}$$

On the basis of equation (2.54),

$$\overline{\left(v_i^p(\vec{x}') - v_i^p(\vec{x}'')\right)^2} = 2\left\{M_{ii}^p(0) - M_{ii}^p(\vec{\rho})\right\} \tag{2.67}$$

from which, with the aid of equation (2.56), there is obtained

$$\left(v_i^p(\vec{x}') - v_i^p(\vec{x}'')\right)^2 = 2\int_{q>p}^{\infty}\left\{1 - e^{i(\vec{q},\ \vec{\rho})}\right\}\Psi_{ik}(q)\,d\Omega \tag{2.67'}$$

Introducing the new nondimensional variables $\alpha = q_1\rho$, $\beta = q_2\rho$, and $\gamma = q_3\rho$ $\left(d\Omega = d\alpha\ d\beta\ d\gamma/\rho^3\right.$ and $(\vec{q},\ \vec{\rho}) = \alpha\xi/\rho + \beta\eta/\rho + \gamma\zeta/\rho$, where ξ, η, and ζ are the projections of $\left.\rho\right)$ and using equation (2.65) result in

$$\overline{\left(v_i^p(\vec{x}') - v_i^p(\vec{x}'')\right)^2} = K^2\rho^{2/3} \tag{2.68}$$

where the constant K^2 is of the order of magnitude of γ (see eq. (2.64)).

A. M. Obukhov (ref. 19) gives an estimate of the value of γ from the fact that the energy of the atmospheric turbulence is derived from the energy of the solar radiation. According to Brent (ref. 21) 2 percent of the sun's energy is converted into the energy of atmospheric turbulence and in this way is dissipated, being converted into heat. This gives $D_0 = 5(\mathrm{erg/sec}^3)$, which leads to the value $\gamma = 2.4$.

All the results given refer to isotropic and homogeneous turbulence. A wind blowing under actual conditions may perhaps be considered as an isotropic turbulence provided all the gigantic air flows in the atmosphere as themselves are not considered turbulence phenomena of the air envelope about the earth.

Such a point of view is possibly justified in meteorology and geophysics, but it is unsuitable for an observer who has little time at his disposal for following the changes in weather (at least in relation to the wind). Hence for short intervals of time in the course of which there is observed a prolonged constancy of the mean wind, it is convenient to consider the turbulence as superimposed on the mean wind (and the change of "mean" wind will lie outside the small scales of time in the course of which the observation is conducted, for example, in the course of minutes or hours). For such an approach the preceding derived equations may be assumed valid in a system of coordinates moving together with the mean wind. The value of the constant γ or K^2 in equation (2.68) may then depend, however, on the absolute magnitude of the mean wind velocity $\vec{v}_0$. This evidently has also been observed in tests (see the following).

11. Fluctuation in Phase of Sound Wave Due to Turbulence of Atmosphere

Very interesting tests on the propagation of sound under the actual conditions of a turbulent atmosphere were conducted by V. A. Krasilnikov (ref. 22). His tests, the main features of which shall be described in this section, are of interest from two points of view. In the first place, they provide a method for the study of atmospheric turbulence; and in the second place, a circumstance which bears a direct relation to our subject, they throw light on the laws of sound propagation in a turbulent atmosphere. They also have a bearing on the accuracy of operation of direction-finding acoustical apparatus.

The test of Krasilnikov consists essentially of the following: At a point Q is placed a sound source (reproducer, fig. 10) at some distance from two microphones M_1 and M_2. The distance $M_1M_2 = l$ is the base of the directional-finding pair. The distance QB from the source of the sound to the center of the base is denoted by L. If the base were turned at a certain angle to QB different from 90°,

then on account of the different distances QM_1 and QM_2 the sound wave would arrive at the microphones M_1 and M_2 with different phase. By determining that position of the base M_1M_2 (by an objective method or by the binaural effect) for which this difference in phase is equal to zero, the direction to the source Q may be determined. On this principle are based acoustical direction finders. Such difference in phase may, however, also be obtained for the "correct" position of the base M_1M_2 (at angle 90° to QB) if the physical conditions of the sound propagation along the two rays QM_1 and QM_2 are different. Such difference in conditions is obtained as a result of the turbulence of the wind.

The velocity of the wind, on which the wave phase depends, is a random function of the point of space. On account of these random differences in the velocity of the wind along the two rays QM_1 and QM_2, the difference in phase of the waves arriving at M_1 and M_2 is likewise a random magnitude. This phase difference ψ was determined in the tests of Krasilnikov; in particular, its mean-square value $\overline{\psi^2}$ was found.

As has been shown (section 7), the phase velocity of sound in the presence of a wind is equal to $V_f = c + v_n$, where c is the velocity of sound and v_n is the projection of the wind velocity on the normal to the wave. In this case the directions of the normals for the rays QM_1 and QM_2 differ little from the direction QB, which is taken for the x-axis. The projection of the wind velocity on this axis is denoted by v, and $V_f = c + v$ is obtained. The phase of the wave passing from Q to M_1 will be

$$\varphi_1 = \omega \int_0^L \frac{dx}{c + v_1} = \varphi_0 - \frac{\omega}{c^2} \int_0^L v_1 dx \tag{2.69}$$

(terms of the order of v_1^2/c^2 and the differences between dx and $ds_1 = dx/\cos\ \theta$ are neglected; see fig. 10) where v_1 denotes the value of the velocity on the ray QM_1. A similar expression will be obtained for the phase in the microphone M_2. For the difference in phase,

$$\psi = \varphi_2 - \varphi_1 = \frac{\omega}{c^2} \int_0^L (v_1 - v_2) dx = \frac{\omega}{c^2} \int_0^L \Delta v\ dx \tag{2.70}$$

where v_2 is the value of the projection of the velocity in the second ray (QM_2) on the axis. The mean value of ψ is, of course, equal to zero. The measure of ψ will be $\overline{\psi^2}$. From equation (2.70),

$$\overline{\psi^2} = \frac{\omega^2}{c^4}\int_0^L dx' \int_0^L dx'' \cdot \overline{\Delta v(x')\Delta v(x'')} \tag{2.71}$$

The averaged magnitude under the integral sign is equal to

$$\begin{aligned}\overline{\Delta v(x')\Delta v(x'')} &= \overline{[v_1(x') - v_2(x')]\ [v_1(x'') - v_2(x'')]} \\ &= \overline{v_1(x')v_1(x'')} + \overline{v_2(x')v_2(x'')} - \overline{v_1(x')v_2(x'')} - \\ &\qquad \overline{v_1(x'')v_2(x')}\end{aligned} \tag{2.72}$$

On the basis of equations (2.57), (2.66), and (2.68),

$$\overline{v_1^2} - \overline{v_1 \cdot v_2} = \frac{1}{2}K^2 r_{12}^{2/3} \tag{2.73}$$

where r_{12} is the distance between the points 1 and 2.

Use is made of equation (2.73) to obtain

$$\overline{\Delta v(x') \cdot \Delta v(x'')} = -\frac{1}{2}K^2\left\{r_{1'1''}^{2/3} + r_{2'2''}^{2/3} - r_{2'1''}^{2/3} - r_{2''1'}^{2/3}\right\} \tag{2.74}$$

from equation (2.72).

In figure 10 it is seen that

$$\left.\begin{aligned} r_{1'1''}^2 &= r_{2'2''}^2 = (x_1 - x_2)^2\ (1 + \theta^2) \qquad (\theta << 1) \\ r_{2'1''}^2 &= r_{2''1'}^2 = (x_1 - x_2)^2 + (x_1 + x_2)^2\ \theta^2 \end{aligned}\right\} \tag{2.75}$$

In this manner there is obtained from equations (2.71), (2.74), and (2.75)

$$\overline{\psi^2} = \left(\frac{\omega}{c^2}\right)^2 K^2 \cdot \int_0^L dx_1 \int_0^L dx_2 \left\{\left[(x_1 - x_2)^2 + (x_1 + x_2)^2 \theta^2\right]^{1/3} - (x_1 - x_2)^{2/3}(1 + \theta^2)^{1/3}\right\} \tag{2.76}$$

Setting $x = x_1/L$ and $y = x_2/L$ gives equation (2.76) in the form

$$\overline{\psi^2} = \left(\frac{\omega}{c^2}\right)^2 K^2 L^{8/3} \theta^{5/3} \int_0^1 \frac{dx}{\theta} \int_0^1 dy \times \left\{\left[\left(\frac{x - y}{\theta}\right)^2 + (x + y)^2\right]^{1/3} - \left(\frac{x - y}{\theta}\right)^{2/3} (1 + \theta^2)^{1/3}\right\} \tag{2.76'}$$

If in the preceding double integral are introduced the variables $\xi = \frac{x - y}{\theta}$ and $\eta = x + y$, then for $\theta \rightarrow 0$ it does not depend on θ and converges to a value of the order of 1. Hence

$$\overline{\psi^2} = \text{constant } K^2 \left(\frac{\omega}{c^2}\right)^2 L^{8/3} \theta^{5/3} \tag{2.77}$$

Denoting the length of the base M_1M_2 by l and remembering that $\theta = l/2L$ result in

$$\sqrt{\overline{\Psi^2}} = \text{constant } K \frac{\omega}{c^2} L^{1/2} l^{5/6} \tag{2.78}$$

Thus, the mean-square fluctuation of phase of the direction finder is proportional to the sound frequency ω, to the square root of the distance from the source, and approximately (exponent 5/6) to the length of the base. The test data of Krasilnikov (loc. cit.) very well confirm both the dependence on ω (the tests were conducted in the range from 1000 to 5000 hertz) and the dependence on l ($\sim l^{5/6}$). It is of interest to remark that the constant K according to the data of Krasilnikov is proportional to the mean velocity of the wind $\vec{v}$. The same result was reached by Gedicke (ref. 23) and Findesen (ref. 24), who measured the turbulence of the atmosphere near the ground. This is in agreement with the remark herein on the fact that the turbulence of the atmosphere, if the observation times under consideration are not too large, must not be considered isotropic (section 10).

The question of the error of the direction finder will now be considered. Let the direction at the source make the angle α with the direction of the base. Then the difference in phase at M_1 and M_2 in the absence of turbulence will be

$$\psi = \frac{2\pi l}{\lambda} \cos \alpha \tag{2.79}$$

The error $\delta\alpha$ in α due to the random fluctuations of ψ will be

$$\delta\alpha = \frac{\lambda}{2\pi l \sin \alpha} \cdot \delta\psi \tag{2.80}$$

At large values of α $(\alpha \sim \pi/2)$ for the mean-root deviation of $\overline{\delta\alpha^2}$ there is obtained

$$\sqrt{\overline{\delta\alpha^2}} = \frac{\lambda}{2\pi l} \sqrt{\overline{\delta\psi^2}} = \frac{\text{constant}}{2\pi} KL^{1/2}l^{-1/6} \tag{2.81}$$

Making use of the data of his tests, Krasilnikov determined the numerical value of the constants entering equation (2.81) as follows:

$$\sqrt{\overline{\delta\alpha^2}} = 0.3 \cdot l^{-1/6} \left(\frac{L}{2\pi}\right)^{1/2} \left(\frac{\bar{v}}{2.7}\right) \tag{2.82}$$

where α is in degrees, l and L in meters, and the mean velocity of the wind is in meters per second. For example, for $l = 1$ meter, $\vec{v} = 2.7$ meters per second, and $L = 2000$ meters, there is obtained $\sqrt{\overline{\delta\alpha^2}} = 3^{\circ}$. The value, if compared with the errors observed in practice of acoustical direction finders, is somewhat exaggerated.

The fact of the matter evidently is that acoustical direction finders generally operate in a range of frequencies of 200 to 500 hertz. For these low frequencies the approximation of the geometrical acoustics on which the preceding computations are based may not be suitable.

Krasilnikov (ibid.) also conducted interesting observations on the random variability of the phase in time. The measurements were in this case conducted with the aid of a single microphone M; the values of the phase ψ at two instants of time separated by a small interval Δt were compared. The results were worked out for the case where the mean wind was perpendicular to the ray joining the source Q and the microphone M (fig. 11). The computation was conducted on the basis of the hypothesis (section 10) on the isotropic and homogeneous character

of the turbulence in a system of coordinates moving together with the wind. In the time interval Δt the phase at the point M changes by

$$\Delta_t \psi = \frac{\omega}{c^2} \int_0^L \Delta v \cdot dx \tag{2.83}$$

where Δv is the change in velocity during the same time. Hence

$$\overline{\Delta_t \psi^2} = \left(\frac{\omega}{c^2}\right)^2 \int_0^L dx' \int_0^L dx'' \cdot \overline{\Delta v(x') \cdot \Delta v(x'')} \tag{2.84}$$

The principal change in the velocity is due to the transport of turbulence by the mean wind so that the change of the velocity v in the time Δt may be represented as the result of the displacement of the turbulence by a small distance $\delta = \vec{v} \cdot \Delta t$. Then

$$\begin{aligned}\overline{\Delta \vec{v}(x')\Delta v(x'')} &= \overline{[v(x', 0) - v(x', \delta)][v(x'', 0) - v(x'', \delta)]} \\ &= \overline{v(x', 0)v(x'', 0)} + \overline{v(x', \delta) \cdot v(x'', \delta)} - \overline{v(x', \delta)v(x'', 0)} \\ &\qquad \overline{v(x', 0)v(x'', \delta)}\end{aligned} \tag{2.85}$$

Making use of the "2/3" law gives

$$\begin{aligned}\overline{\Delta v(x')\Delta v(x'')} &= -\frac{K^2}{2}\left\{2(x' - x'')^{2/3} - 2[(x' - x'')^2 + \delta^2]^{1/3}\right\} \\ &= K^2\left\{(r^2 + \delta^2)^{1/3} - r^{2/3}\right\};\ r^2 = (x' - x'')^2\end{aligned} \tag{2.86}$$

Substituting equation (2.86) in equation (2.84) and applying to the obtained double integral the same considerations that were applied to the integral (2.76) result in

$$\overline{\Delta_t \psi^2} = \text{constant } K^2 L^{8/3} \left(\frac{\omega}{c^2}\right)^2 \left(\frac{\delta}{L}\right)^{5/3} \tag{2.87}$$

where the constant is found to be $\cong 3$. Thus

$$\sqrt{\overline{\Delta_t \psi^2}} = K\sqrt{3}\, L^{4/3} \frac{\omega}{c^2} (\vec{v} \cdot \Delta t)^{5/6} \tag{2.88}$$

Test data give the relation $(\vec{v} \cdot \Delta t)^{4/6}$ rather than $(\vec{v} \cdot \Delta t)^{5/6}$. It is as yet difficult to explain the source of this divergence. Equation (2.88), since $\overline{\Delta_t\psi^2}$, L, $\vec{v}$, Δt, and ω are known from tests, permits determining the constant K in the "2/3" law. For $\vec{v} = 6.5$ meters per second there is obtained from tests

$$K = 11(\mathrm{cm}^{2/3}/\mathrm{sec})$$

The turbulence measurements at the height of 2 meters above the earth conducted by A. M. Obukhov and N. D. Ershova give (for $\vec{v} = 3$ m/sec) the value $K = 3.1$ centimeters$^{2/3}$ per second.

Gedicke (ref. 23) obtains for K (at $\vec{v} = 0.65$ m/sec and height 1.15 m) the value 2.05 centimeters$^{2/3}$ per second. It follows that the order of magnitude of K is in all cases obtained as that of unity. The increase of the constant K with the velocity of the wind is a fact, however, which shall have to be taken into consideration in another connection.

12. Dissipation of Sound in Turbulent Flow

It is a well-known experimental fact that in the presence of wind the audibility of sounds is markedly decreased. This decrease in audibility is not a consequence of the curvature of the rays in a wind with velocity gradient considered in sections 8 and 9; it has a more complicated character and is connected with the turbulence of the wind. The first to point out these phenomena in connection with the occurrence of acoustical fading were Dahl and Devick (ref. 25). The same phenomenon of acoustical fading was investigated by Y. M. Sukharevskii in measurements on mountains (Elbruz expedition of the USSR Academy of Sciences, 1940). The general impairment of audibility in a wind has also been pointed out by Stewart (ref. 26).

From the experimental viewpoint the problem was investigated most thoroughly by Sieg (ref. 27) who showed the existence in a wind of an additional damping of sound exceeding the damping associated with the molecular properties of the gas (viscosity, heat conductivity, and Kneser effect). The results of Sieg may be essentially reduced to the following: In the frequency interval 250 to 4000 hertz in a weak wind (1 to 2 m/sec or at an almost complete calm) considerable fluctuations in the sound intensity (fading) are not observed, but the intensity of

the sound drops with increasing distance, the damping coefficient α being equal to 1.5 to 2.2 decibels at 100 meters[8]. Sieg does not find any dependence of the coefficient α on the frequency. It should be borne in mind, however, that the accuracy of Sieg's observations is not large; the directional characteristics of the source were not taken into account, and the conditions under which the points for the various frequencies were taken were not identical. For this reason this result does not appear entirely reliable; it gives rather the order of magnitude of α which in the interval 250 to 4000 hertz does not change.

In the case of a strong gusty wind the coefficient of damping decreases, reaching a magnitude of 5 to 9 decibels at 100 meters (for a wind with gusts of 7 to 17 m/sec). Under these conditions the dependence of α on the frequency becomes more marked, α being equal to 5 decibels for 250 hertz, 8 decibels for 2000 hertz, and 9 decibels for 4000 hertz (at 100 m). Under the same conditions, fading is observed; the fluctuations of the intensity attain 25 decibels. Both these effects are explained without forcing by the theory of the propagation of sound in a turbulent flow (refs. 28 and 29). In considering the propagation of sound in a turbulent flow, it is first of all necessary to bear in mind that those fluctuations of the velocity of the stream having the scale l which is considerably greater than the length of the sound wave λ do not lead to the dissipation of the sound. They bring about only changes in the shape of the rays and therefore a general fluctuation of the sound intensity at the location of the receiver (fading). The effect of these large-scale pulsations may be considered by the method of geometrical acoustics. Hence the velocity of a turbulent flow must be decomposed into two components $\vec{v}$ (macrocomponent) and $\vec{u}$ (microcomponent):

$$\begin{aligned} \vec{v} &= \int_{q<q_0} e^{i(\vec{q},\vec{x})}\, \vec{U}(d\Omega(\vec{q})) \\ \vec{u} &= \int_{q<q_0} e^{i(\vec{q},\vec{x})}\, U(d\Omega(\vec{q})) \end{aligned} \tag{2.89}$$

where $\vec{v}$ includes the mean velocity of the flow $\vec{v}_0$. The magnitude $q_0 = k/\mu$, where $k = 2\pi/\lambda$, is the wave number of the sound wave and μ is a nondimensional number $\gg 1$. The dissipation of sound from a parallelepiped L^3 where $L \gg \lambda$ and $L \lesssim 2\pi/q_0$ will now be considered.

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[8]There is here subtracted the molecular absorption (Kneser effect with account taken of the humidity of the air). It has a considerable value starting with frequencies of 1000 hertz. The classical absorption due to the viscosity and the heat conductivity is of significance only for frequencies greater than 10,000 hertz.

Under this condition the velocity $\vec{v}$ may be considered approximately constant in the volume.

In a local system of coordinates which move with the velocity $\vec{v}$, the frequency of the sound f varies in it (Doppler effect) only by the small amount $f \cdot v/c$, but the frequencies of the turbulent fluctuations in this system are equal to $\nu = u(l)/l$, where l is the scale of the pulsations and $u(l)$ is the velocity of the pulsations associated with this scale. According to the "2/3" law, $u^2 = \text{constant} \cdot l^{2/3} \ll f$ (where constant $\sim 1\ \text{cm}^{4/3}/\text{sec}$), so that $\nu \cong \text{constant}^{1/2} \cdot l^{-2/3} \ll f$ for all values of f of practical applications.[9] Hence in the propagation of sound through a turbulent flow, only the instantaneous picture of the turbulence and not its process with time is of significance. For the same reason it is not to be supposed that the damping of sound in a turbulent flow is conditioned by the existence of turbulent viscosity. The tensor of the turbulent stresses with which the concept of turbulent viscosity is associated is obtained as a result of the averaging of the turbulent pulsations for the given mean flow. This averaging presupposes that all the changes in the mean flow occur more slowly than the random pulsations of velocity produced by the turbulence. For a sound wave the situation is the reverse $(\nu \ll f)$. The effect of the turbulent flow on the sound wave should reduce to the dissipation of sound in a manner similar to the dissipation of light passing through a turbid medium; in both cases random changes of the velocity of the wave propagation occur. An estimate of the magnitude of this dissipation is now made. A start will be made from the equation of A. M. Obukhov, approximately taking into account the presence of vortices. The quasipotential of the sound waves is denoted by ψ and the total velocity of the flow by $\vec{V} = \vec{v} + \vec{u}$ to obtain from equation (2.84) (for $\nabla\Pi_0 = 0$, $\nabla \log c^2 = 0$, $v/c \ll 1$)

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi + \frac{2}{c^2}\left(\vec{V}, \nabla\frac{\partial\psi}{\partial t}\right) - \int^t (\nabla\psi, \Delta\vec{V})dt = 0 \qquad (2.90)$$

Passing over to a local system of coordinates in which $v = 0$ results in

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[9] It should be remarked that there exists a minimum scale of turbulence $l = l_{min} = 1/\sqrt{\varkappa} \cdot \sqrt[4]{\mu^3/D_0\rho^3}$ (D_0 is the supply of energy, μ is the viscosity of the medium, ρ is its density, and $\varkappa$ a number $\cong 1$. See A. M. Obukhov (ref. 19). On account of this, the inequality $\nu \ll f$ may be violated only for f of the order of several hertz.

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi = -\frac{2}{c^2}\left(\vec{u}, \nabla\frac{\partial\psi}{\partial t}\right) + \int^t (\nabla\psi, \Delta\vec{u})dt \tag{2.91}$$

The right side of this equation will be considered as the disturbance. By rejecting it completely, the zeroth approximation ψ_0, representing the fundamental wave, is obtained as

$$\psi_0 = Ae^{i[\omega t-k(\vec{n}_1,\vec{x})]} \tag{2.92}$$

where $\vec{n}_1$ is the unit vector in the direction of propagation of the fundamental wave $k = \omega/c$. The complete solution will be

$$\psi = \psi_0 + \varphi \tag{2.93}$$

where φ is the dissipated wave. For large distances R from the parallelepiped considered, φ is of the form

$$\varphi = \frac{B}{R}\, e^{i(\omega t-kR)} \tag{2.94}$$

The amplitude of the dissipated wave B is determined by use of the method of the theory of disturbances and the substitution of φ_0 in the right side of equation (2.91) in place of ψ. There is then obtained

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = -\frac{2}{c^2}\left(\vec{u}, \nabla\frac{\partial\psi_0}{\partial t}\right) + \int_0^t (\nabla\psi_0, \Delta\vec{u})dt = Q \tag{2.95}$$

The solution of the wave equation (2.95) having the form of equation (2.94), as is known, is equal to

$$\varphi(\vec{x}, t) = -\frac{1}{4\pi}\int_{L^3}\frac{Q\left(\vec{x}', t - \frac{r}{c}\right)}{r}dv' \tag{2.96}$$

where $dv' = dx' \cdot dy' \cdot dz'$ and r is the distance between the points $\vec{x}$ (point of observation) and $\vec{x}'$ (source of dissipated wave). Let $\vec{n}$ be the unit vector in the direction of the dissipated ray (fig. 12), R the distance from the center of the parallelepiped, and θ the angle of dissipation (angle between $\vec{n}_1$ and $\vec{n}$). Then, as follows from the sketch, $r = R - (\vec{x}', \vec{n})$ (neglecting terms of the order of x'/R). Substituting in equation (2.96) Q from equation (2.95) and using equation (2.92) give for $R \to \infty$

$$\varphi = -\frac{1}{4\pi} \cdot \frac{e^{i(\omega t - kR)}}{R} \cdot \frac{A}{c} \int_{L^3} (2\vec{u}'k^2 + \Delta\vec{u}', \vec{n}_1) e^{i(\vec{K},\vec{x}')} dv' \qquad (2.97)$$

where the vector $\vec{K}$ is equal to

$$\vec{K} = k(\vec{n} - \vec{n}_1); \qquad K = 2k \sin\frac{\theta}{2} \qquad (2.98)$$

and $\vec{u}'$ is the value of the velocity $\vec{u}$ at the point $\vec{x}'$. Thus the amplitude of the dissipated wave B is equal to

$$B = \frac{A}{4\pi c} \int_{L^3} (2\vec{u}k^2 + \Delta\vec{u}, \vec{n}_1) e^{i(\vec{K},\vec{x}')} dv' \qquad (2.99)$$

The coefficient of damping α is expressed in terms of the amplitude of the dissipated wave. The flow of sound energy N into the base of the parallelepiped L^2 is proportional to A^2L^2, while the flow of energy dissipated from the parallelpiped is obtained by integration over a distant sphere of radius R and is proportional to $R^2 \int |B|^2 d\Omega$, where $d\Omega$ denotes integration over all the directions of dissipation. Since interest lies not in the instantaneous value of the dissipation but in the mean value, $R^2 \int \overline{|B|^2} d\Omega$ must be taken in place of the previous expression, where the bar over $|B|^2$ denotes the averaging over the velocity fluctuations of the turbulent flow. The mean decrease of the energy flow in passing through the parallelepiped L^3 will be

$$\Delta N = \alpha N L \qquad (2.100)$$

from which $\alpha = \Delta N/NL$, and since $\Delta N = \beta \cdot R^2 \int |B|^2 d\Omega$ (β is the factor of proportionality) and $N = \beta A^2 L^2$,

$$\alpha = \frac{\int \overline{|B|^2} d\Omega}{A^2 L^3} \qquad (2.101)$$

From equation (2.99) it follows that

$$\overline{|B|^2} = \frac{A^2}{16\pi^2 c^2} \int_{L^3} dv' \int_{L^3} dv'' e^{i(\vec{K}\vec{\rho})} \times$$

$$\overline{\left\{(2u_1' k^2 + \Delta u_1')(2u_1'' k^2 + \Delta u_1'')\right\}} \tag{2.102}$$

where $\vec{\rho} = \vec{x}'' - \vec{x}'$ is the radius vector between the points $\vec{x}''$ and $\vec{x}'$, and u_1 is the projection of $\vec{u}$ on $\vec{n}_1$. Introducing in place of $\vec{x}'$ and $\vec{x}''$ the relative coordinates $\vec{\rho}$ and the coordinates of the center of gravity $\vec{x} = \dfrac{\vec{x}' + \vec{x}''}{2}$ results in

$$\overline{|B|^2} = \frac{A^3 L^3}{16\pi^2 c^3} \int dv_\rho e^{i(\vec{K},\ \rho)} \times$$

$$\left\{4k^4 M_{11}(\vec{\rho}) + 4k^2 \Delta M_{11}(\vec{\rho}) + \Delta^2 M_{11}(\vec{\rho})\right\} \tag{2.103}$$

where

$$M_{11}(\vec{\rho}) = \overline{u_1' \cdot u_1''} = \overline{u_1(\vec{x}')u_1(\vec{x}'')}$$

is the moment of correlation. This moment is identical with the moment $M_{ik}^{p}(\vec{\rho})$ introduced in section 10 (see eq. (2.54)) for $i = k = 1$ and $p = q_0$. Now equations (2.56) and (2.65) are used to find that

$$M_{11}(\rho) = \int_{q<q_0} e^{-i(\vec{q},\rho)} \left(1 - \frac{q_1^2}{q^2}\right) \gamma q^{-11/3} dq_1 dq_2 dq_3 \tag{2.104}$$

The multiplication of the expression under the integral in equation (2.104) by $-q^2$ and by q^4, respectively, is obtained by simply applying to $M_{11}(\vec{\rho})$ the operators Δ and Δ^2. Substitution of the moment (2.104) in equation (2.103) leads to integrals of the form

$$\int dv_\rho \int_{q>q_0} dq_1 \cdot dq_2 \cdot dq_3 \cdot e^{i(\vec{K}-\vec{q},\rho)} F(\vec{q})$$

$$= (2\pi)^3 \int_{q>q_0} dq_1 dq_2 dq_3 \cdot \delta(K_1 - q_1)\delta(K_2 - q_2) \times$$

$$\delta(K_3 - q_3) \cdot F(\vec{q}) = (2\pi)^3 F(\vec{K}) \qquad \text{for } K > q_0$$

$$= 0 \qquad \text{for } K < q_0 \qquad (2.105)$$

Here $\delta(x)$ is the symbol of the δ-function (see section 6). Hence $\overline{|B|^2}$ is obtained as

$$\overline{|B|^2} = \frac{2\pi A^2 L^3 k^4}{c^2} \left(1 - \frac{K^2}{k^2} + \frac{K^4}{4k^4}\right)\left(1 - \frac{K_1^2}{K^2}\right)\gamma \cdot K^{-11/3} \qquad (2.106)$$

(for $K = q_0$, otherwise $\overline{|B|^2} = 0$). From this, on the basis of equation (2.101),

$$\alpha = \frac{2\pi k^4}{c^2} \int \left(1 - \frac{K^2}{k^2} + \frac{K^4}{4k^4}\right)\left(1 - \frac{K_1^2}{K^2}\right)\gamma K^{-11/3}\, d\Omega \qquad (2.107)$$

where the integration over the angles is extended to the values $K > q_0$. Setting $\sin \theta/2 = \xi$ and $d\Omega = \sin\theta\, d\theta\, d\varphi = 4\xi \cdot d\xi \cdot d\varphi$ shows that the integration over $\xi = K/2k$ is extended from $\xi = 1/2\mu$ to $\xi = 1$. Carrying out this elementary integration yields

$$\alpha = \mu^{5/3}\, \beta \left(\frac{2\pi\gamma^{1/2}\lambda^{1/3}}{c}\right)^2 \cdot \frac{1}{\lambda} \qquad (2.108)$$

where

$$\beta = \frac{3}{5}(2\pi)^{1/3}\left\{1 + 25(2\mu)^{-1/3} - 21(2\mu)^{-4/3} + 0(\mu^{-4})\right\} \qquad (2.109)$$

The magnitude $2\pi\gamma^{1/2}\lambda^{1/3}$ is the velocity of the turbulent pulsations, the scale of which is less than λ. Thus the coefficient of damping of the sound waves in a turbulent flow is proportional to the square of the Mach number $(M_a = u(\lambda)/c)$ for the velocity of the turbulent pulsations of scale less than λ and inversely proportional to the length of

the sound wave λ. The magnitude $2\pi\gamma^{1/2}$, on the basis of the estimate of A. M. Obukhov given in section 10, is equal to 3. The data of V. A. Krasilnikov (section 11) and also of A. M. Obukhov and N. D. Ershov (section 11) give, for a moderate wind, $2\pi\gamma^{1/2} \cong 6$. As already pointed out, the turbulence of the wind must not be considered isotropic so that, in general, $2\pi\gamma^{1/2}$ is an increasing function of the wind velocity. If use is made of the as yet not very reliable test data presented in section 10, it is necessary to assume γ proportional to the wind velocity. This explains the increase in the coefficient of damping α with the wind velocity. The dependence of the coefficient α on the length of the sound wave is obtained in the form $\lambda^{-1/3}$, that is, a very weak dependence; but, on the basis of what has been said, this dependence does not contradict the test data of H. Sieg. In order to estimate the value of the numerical factor μ, use is again made of Sieg's data for a weak wind. In this case $2\pi\gamma^{1/2} \cong 6$. The coefficient α is equal to 1.5 decibels in 100 meters, which in absolute units gives $\alpha = 10^{-5}$centimeters^{-1}. For $f = 500$ hertz ($\lambda = 68$ cm) there is obtained $\mu \cong 10$. This value of μ should be considered as entirely reasonable.

13. Sound Propagation in Medium of Complex Composition,

Particular in Salty Sea Water

In the theory of sound propagation presented, the medium was assumed homogeneous in its composition. In practice, however, it is necessary to deal with cases where the composition of the medium varies from point to point (air, for example, the humidity of which is different at different places or sea water with variable saltiness).

All the theorems of geometrical acoustics that were derived in sections 7, 8, and 9 retain their validity for media of variable composition.[10] The initial general equations of the acoustics of a nonhomogeneous and moving medium must, however, be modified.

The need for modifying these equations is dictated by the fact that in a medium of complex composition the pressure p depends not only on the density of the medium ρ and the entropy S but also on the concentrations C_k of the individual components forming the medium (for example, on the concentration of the water vapor in the air, the concentration of salt dissolved in the water, and so forth). Hence the equation of state must be written not in the form $p = Z(\rho, S')$, as previously, but in the form

[10]Provided, of course, that the fundamental hypothesis of geometrical acoustics on the smoothness of all changes in state of the medium is not violated.

$$p = Z(\rho, S, C) \tag{2.110}$$

Here ρ is the density of the medium and C is the concentration of the second component in it; $C = \rho''/\rho'$, where ρ'' is the density of the dissolved component, and ρ' is the density of the solvent ($\rho = \rho' + \rho'' = \rho'(1 + C)$).

Further, to the hydrodynamic equations it is necessary to add equations governing the changes in concentration of the dissolved component. These changes are produced by convection, diffusion, and the action of the gravitational force. In order to write down the corresponding equations, the flow of the dissolved component $\vec{J}''$ is noted as

$$\vec{J}'' = \vec{v}\rho' C + \vec{i} \tag{2.111}$$

$$\vec{i} = -\rho' D_1 \nabla C - \rho' D_2 \nabla T + \rho' u\vec{g} C \tag{2.111'}$$

where D_1 is the coefficient of diffusion, D_2 is the coefficient of thermodiffusion, u is the mobility of the solvent in the field of gravity, and $\vec{g}$ is the acceleration of gravity. The first term in equation (2.111) $\vec{v}\rho' C$ represents the part of the flow due to the convection of the substance, and the second term $\vec{i}$, the part of the flow due to the irreversible processes (diffusion, thermodiffusion, and motion in the gravity field with friction). On the basis of the law of conservation of matter,

$$\frac{\partial(\rho' C)}{\partial t} + \operatorname{div} \vec{J}'' = 0 \tag{2.112}$$

The density of the pure medium ρ' is subject, of course, to the equation of continuity

$$\frac{\partial \rho'}{\partial t} + \operatorname{div}(\rho' \vec{v}) = 0 \tag{2.113}$$

The required equation for C is obtained from equations (2.112) and (2.113):

$$\frac{\partial C}{\partial t} + (\vec{v} \nabla C) = -\frac{1}{\rho} \operatorname{div} \vec{i} \tag{2.114}$$

For the total density $\rho = \rho'(1 + C)$ there is obtained from equations (2.112) and (2.113)

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = - \operatorname{div}\vec{i} \tag{2.115}$$

The fundamental dynamic equation of hydrodynamics

$$\frac{\partial\vec{v}}{\partial t} + [\operatorname{rot}\vec{v}, \vec{v}] + \nabla\frac{v^2}{2}$$

$$= - \frac{\nabla p}{\rho} + \vec{g} + \nu\,\Delta\vec{v} + \frac{\nu}{3}\,\nabla\operatorname{div}\vec{v} \tag{2.116}$$

remains unchanged. The equation of entropy will be written in the abbreviated form

$$\frac{\partial S}{\partial t} + \overline{(\vec{v}\nabla S)} = \psi \tag{2.117}$$

where ψ denotes the changes in entropy due to the irreversible processes occurring in the motion of the fluid (ψ contains terms proportional to ν, λ, D_1, D_2, and u) and also the possible supply of heat from without.

Equations (2.110), (2.114), (2.115), (2.116), and (2.117) form a complete system of equations for a medium in which some component is dissolved (water vapor in air, salt in water, and so forth).

In the propagation of sound all the magnitudes characterizing the medium receive small increments so that $\vec{v}$ is replaced by $\vec{v} + \vec{\xi}$, p by $p + \pi$, ρ by $\rho + \delta$, S by $S + \sigma$, and C by $C + \Sigma$, where Σ denotes a small change in concentration of the dissolved component that occurs in the medium on the passage of a sound wave. Substituting these changed values in equations (2.110), (2.114), (2.115), (2.116), and (2.117), restricting to a linear approximation, and rejecting the added terms proportional to ν, λ, D_1, D_2, and u, that is, leaving aside the irreversible processes accompanying the sound wave, give[11]

$$\frac{\partial\vec{\xi}}{\partial t} + [\operatorname{rot}\vec{v}, \vec{\xi}] + [\operatorname{rot}\vec{\xi}, \vec{v}] + \nabla(\vec{v}, \vec{\xi}) = - \frac{\nabla\pi}{\rho} + \frac{\nabla p\delta}{\rho^2} \tag{2.118}$$

$$\frac{\partial\delta}{\partial t} + (\vec{v}, \nabla\delta) + (\vec{\xi}, \nabla\rho) + \rho\operatorname{div}\vec{\xi} + \delta\operatorname{div}\vec{v} = 0 \tag{2.119}$$

$$\frac{\partial\sigma}{\partial t} + (\vec{v}, \nabla\sigma) + (\vec{\xi}, \nabla S) = 0 \tag{2.120}$$

[11] The diffusion of the salt may give an absorption of sound in addition to that due to the viscosity and heat conductivity.

$$\frac{\partial \Sigma}{\partial t} + (\vec{v}, \nabla\Sigma) + (\vec{\xi}, \nabla C) = 0 \qquad (2.121)$$

$$\pi = c^2 \delta + h\sigma + g\Sigma \qquad (2.122)$$

where

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_{S,C}, \quad h = \left(\frac{\partial p}{\partial S}\right)_{\rho,C}, \quad g = \left(\frac{\partial p}{\partial C}\right)_{\rho,S} \qquad (2.123)$$

The square of the adiabatic velocity of sound for constant concentration of the solution is c^2.

These equations must be considered as the fundamental equations for the propagation of sound in a nonhomogeneous and moving medium of variable composition. If by C there is understood the concentration of the water vapors in the air, these will be the equations for the propagation of sound in a humid atmosphere.

The same equations may also be considered as the equations for sound waves propagated in salty sea water. For this, C must be considered as the concentration of the salt dissolved in the water. In the presence of entropy gradients $(\nabla S \neq 0)$, as in the presence of gradients of the concentration of the dissolved component $(\nabla C \neq 0)$, the right side of equation (2.118) is not a total differential of some function. Hence even in the absence of vorticity (i.e., for rot $\vec{v} = 0$) the sound will be vortical (rot $\vec{\xi} \neq 0$). Because of this the system of equations (2.118) to (2.122) cannot be reduced to an equation for a single function (for example, to an equation for the sound potential, to an equation for the sound pressure, and so forth).

In order to change to the equations of geometrical acoustics it is noted that equation (2.121) does not differ formally from equation (2.120). Hence, following the same method which was used in section 7 for deriving the equations of the geometric acoustics of a medium of constant composition, and assuming, in addition to equations (2.5) and (2.7),

$$\Sigma = \Sigma_0 \cdot e^{i\Phi}; \qquad \Sigma_0 = \Sigma_0' + \frac{\Sigma_0''}{ik_0} + \cdot \cdot \cdot \qquad (2.124)$$

result in

$$\sigma_0' = 0$$

$$\Sigma_0' = 0 \qquad (2.125)$$

that is, in the first approximation of geometric acoustics the sound is propagated not only isentropically but leaves unchanged the composition of the medium ($\Sigma_0' = 0$). All the remaining conclusions with regard to geometric acoustics previously obtained likewise remain in full force. The effect of the nonhomogeneity of composition of the medium is in this approximation reduced to the effect on the velocity of sound in the medium c and on the density of the medium ρ.

The sound will be propagated within the ray tubes with velocity

$$\vec{V}_s = c\vec{n} + \vec{v}; \qquad c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_{S,C}} \tag{2.126}$$

and the pressure π will be subject to the law

$$\frac{\pi^2 s}{\rho q c^2} = \text{constant} \tag{2.127}$$

(compare section 7, eq. (2.32)).

The particular case when the medium is at rest is now considered. This case is of special interest for water in which the velocity of sound is large while the velocity of flow is small.

For a medium at rest ($\vec{v} = 0$), from equations (2.118), (2.119), (2.120), (2.121), and (2.122),

$$\frac{\partial \vec{\xi}}{\partial t} = -\frac{\nabla\pi}{\rho} + \frac{\nabla p}{\rho^2}\left(\frac{\pi - h\sigma - g\Sigma}{c^2}\right) \tag{2.118'}$$

$$\frac{\partial}{\partial t}\left(\frac{\pi - h\sigma - g\Sigma}{c^2}\right) + \rho \operatorname{div}\vec{\xi} = 0 \tag{2.119'}$$

$$\frac{\partial \sigma}{\partial t} = -(\vec{\xi}, \nabla S) \tag{2.120'}$$

$$\frac{\partial \Sigma}{\partial t} = -(\vec{\xi}, \nabla C) \tag{2.121'}$$

Setting $\pi/\rho = \Pi$ and making use of equations (2.120') and (2.121') give the equations for Π and $\vec{\xi}$:

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = -\nabla\frac{\partial \Pi}{\partial t} + \frac{\nabla p'}{\rho c^2} \cdot \frac{\partial \Pi}{\partial t} + \frac{\nabla p}{\rho^2 c^2}(\nabla p', \vec{\xi}) \tag{2.128}$$

$$\frac{1}{c^2}\frac{\partial \Pi}{\partial t} + \operatorname{div}\vec{\xi} + \frac{(\nabla p', \vec{\xi})}{\rho c^2} = 0 \tag{2.129}$$

where

$$\nabla p' = h\nabla S + g \cdot \nabla C = \nabla p - c^2 \nabla \rho \tag{2.130}$$

Substituting $\partial\Pi/\partial t$ from equation (2.129) in equation (2.128) gives the equation for the velocity of the sound vibrations

$$\frac{\partial^2 \vec{\xi}}{\partial t^2} = \nabla\left(c^2 \cdot \operatorname{div}\vec{\xi} + \frac{(\nabla p', \vec{\xi})}{\rho}\right) - \frac{\nabla p' \cdot \operatorname{div}\vec{\xi}}{\rho} + \frac{\nabla \rho}{\rho} \cdot \frac{(\nabla p' \vec{\xi})}{\rho} \tag{2.131}$$

This is the equation for the propagation of sound vibrations in a medium at rest in which the density, temperature (entropy), and concentration of the dissolved substance vary. It is seen from the equation that for the computation of $\vec{\xi}$ it is sufficient to know c, p, and ρ as point functions, where c is the adiabatic velocity of sound and ρ is the total density of the medium.

Equation (2.131) does not reduce to an equation for the potential or the pressure.

After $\vec{\xi}$ has been found from equation (2.131), the sound pressure is found from equation (2.129) as

$$\frac{\pi}{\rho} = \Pi = \int^{t}\left\{c^2 \operatorname{div}\vec{\xi} + \frac{(\nabla p', \vec{\xi})}{\rho}\right\} dt \tag{2.132}$$

In certain special cases equation (2.131) may approximately be replaced by the simpler wave equation. In fact, a medium for which the term in equation (2.121) containing ∇c^2 is much greater than the terms containing $\nabla p'$ is assumed. Then, rejecting the terms with $\nabla p'$ and setting $\vec{\xi} = -\nabla\varphi$ (φ is the velocity potential of the sound vibrations), the usual wave equation is obtained:

$$\frac{\partial^2 \varphi}{\partial t^2} = c^2 \Delta\varphi \tag{2.133}$$

in which, however, c varies from point to point.

The term with ∇c^2 is $\nabla c^2 \operatorname{div} \vec{\xi}$ and in order of magnitude is equal to $\nabla c^2 \cdot k\vec{\xi}$ (k is the wave number). The greatest term containing $\nabla p'$ is $\nabla p' \cdot \operatorname{div} \vec{\xi}/\rho$, in order of magnitude equal to $\nabla p' \cdot k\vec{\xi}/\rho$. Hence the terms containing $\nabla p'$ may be rejected and the term containing ∇c^2 retained if

$$\nabla c^2 \gg \frac{\nabla p}{\rho} \tag{2.134}$$

In order to obtain the condition satisfying this inequality, c^2 and p' are considered as functions of p, T, and C. Then

$$\nabla c^2 = \left(\frac{\partial c^2}{\partial p}\right)_{T,C} \cdot \nabla p + \left(\frac{\partial c^2}{\partial T}\right)_{p,C} \nabla T + \left(\frac{\partial c^2}{\partial C}\right)_{p,T} \cdot \nabla C \tag{2.135}$$

$$\frac{\nabla p'}{\rho} = \frac{\nabla p}{\rho} - \frac{c^2}{\rho} \cdot \left\{\left(\frac{\partial \rho}{\partial p}\right)_{T,C} \cdot \nabla p + \left(\frac{\partial \rho}{\partial T}\right)_{p,C} \cdot \nabla T + \left(\frac{\partial \rho}{\partial C}\right)_{T,p} \cdot \nabla C\right\} \tag{2.136}$$

Here $(\partial\rho/\partial p)_{p,C} = 1/a^2$ (a^2 is the square of the isothermal velocity of sound), $(\partial\rho/\partial T)_{p,C} = -\rho\beta$ (β is the coefficient of volume expansion), and $(\partial\rho/\partial C)_{T,p} = -\rho\varkappa$, where $\varkappa = \frac{1}{V}\left(\frac{\partial V}{\partial C}\right)_{T,p}$ is the relative change of volume of the fluid (gas) with change in the concentration of salt (or vapor, respectively).

Since

$$a^2 = \frac{c_p}{c_v} \cdot c^2$$

and

$$c_p - c_v = a^2\beta^2 T$$

from equation (2.136)

$$\frac{\nabla p'}{\rho} = -\frac{a^2\beta^2 T}{\rho \cdot c_v} \cdot \nabla p + c^2\beta \cdot \nabla T + c^2\varkappa \nabla C \tag{2.137}$$

These equations, on the basis of experimental data, permit solving the problem of satisfying (or not satisfying) inequality (2.134).

In particular, for salt sea water, this inequality is evidently satisfied. In fact, for water $\beta = 2 \cdot 10^{-4}$ at 18° C, and at 4° C, $\beta = 0$. The magnitude $\varkappa = \frac{1}{V} (\partial V/\partial C)_{p,T}$ for a solution of NaCl or KCl at 15° is about 0.15 to 0.20. According to the measurements of A. Wood (refs. 30 and 31), the velocity of sound in sea water at $t = 16.95^{\circ}$ and saltiness of 35.02 percent (that is, at $C = 3.5 \cdot 10^{-2}$) is equal to 1526.3±0.3 meters per second and is governed by the equation

$$c = 1450 + 4.206t - 0.0366t^2 + 1.137 \cdot 10^3(C - 3.5 \cdot 10^{-2})$$

whence

$$(\partial c^2/\partial C) = 2c \cdot 1.137 \cdot 10^3 = 1.42 \cdot c^2$$

It is seen that $\partial c^2/\partial C \gg \varkappa c^2$. Further, $(\partial c^2/\partial T)_{p,C} = 2c \cdot 4.2 = 5.8 \cdot 10^{-3} \cdot c^2$ and $\beta c^2 = 2 \cdot 10^{-4} \cdot c^2$, that is, $(\partial c^2/\partial T)_{p,C} \gg \beta \cdot c^2$.

Thus the magnitude ∇c^2 for salt sea water considerably exceeds the magnitude $\nabla p'/\rho$. Hence the wave equation (2.133) may be assumed to describe the propagation of sound in calm sea water in an entirely satisfactory manner.

CHAPTER III

MOVING SOUND SOURCE

14. Wave Equation in an Arbitrarily Moving System of Coordinates

In a system of coordinates (x,y,z,t) associated with the air at rest, the wave equation for the acoustic potential φ is

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0; \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{3.1}$$

It is assumed that the position of a moving source of sound is determined by the coordinates

$$\left.\begin{aligned} x &= X(t) \\ y &= Y(t) \\ z &= Z(t) \end{aligned}\right\} \tag{3.2}$$

In this case it is convenient to introduce a system of coordinates (ξ,η,ζ,τ) connected with the sound source

$$\begin{aligned} \xi &= x - X(t) \qquad & \eta &= y - Y(t) \\ \zeta &= z - Z(t) \qquad & \tau &= t \end{aligned} \tag{3.3}$$

In this system of coordinates the velocity of a wind $\vec{V}_0$ has the components

$$\left.\begin{aligned} V_{0x} &= -\frac{dX}{dt} = -v_x \\ V_{0y} &= -\frac{dY}{dt} = -v_y \\ V_{0z} &= -\frac{dZ}{dt} = -v_z \end{aligned}\right\} \tag{3.4}$$

Equation (3.1) is then transformed to the system of coordinates (ξ,η,ζ,τ). For this purpose

$$\varphi(x,y,z,t) = \varphi(\xi + X(\tau), \eta + Y(\tau), \zeta + Z(\tau); \tau) \tag{3.5}$$

so that

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial x} = \frac{\partial\varphi}{\partial\xi}, \ \frac{\partial\varphi}{\partial y} = \frac{\partial\varphi}{\partial\eta}, \ \frac{\partial\varphi}{\partial z} = \frac{\partial\varphi}{\partial\zeta}\\ \nabla_{xy}\varphi = \nabla_{\xi\eta\zeta}\varphi = \nabla\varphi\end{aligned}\right\} \tag{3.6}$$

that is,

$$\left.\begin{aligned}\frac{\partial\varphi}{\partial t} &= \frac{\partial\varphi}{\partial\tau} - (\vec{v},\nabla)\varphi\\ \frac{\partial^2\varphi}{\partial t^2} &= \frac{\partial^2\varphi}{\partial\tau^2} - 2(\vec{v},\nabla)\frac{\partial\varphi}{\partial\tau} + (\vec{v},\nabla)(\vec{v},\nabla)\varphi - \left(\frac{d\vec{v}}{d\tau},\nabla\right)\varphi\end{aligned}\right\} \tag{3.6'}$$

Hence, the wave equation (3.1) in the system of coordinates ξ, η, ζ will be

$$\left.\begin{aligned}\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial\tau^2} + \frac{2}{c^2}(\vec{v},\nabla)\frac{\partial\varphi}{\partial\tau} - \\ \frac{1}{c^2}(\vec{v},\nabla)(\vec{v},\nabla)\varphi + \frac{1}{c^2}\left(\frac{d\vec{v}}{dt},\nabla\right)\varphi = 0\end{aligned}\right\} \tag{3.7}$$

or, if in place of the velocity of the source $\vec{v}$, the velocity of the wind V_0 is introduced, then

$$\left.\begin{aligned}\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial\tau^2} - \frac{2}{c^2}(\vec{V}_0,\nabla)\frac{\partial\varphi}{\partial t} - \frac{1}{c^2}(\vec{V}_0,\nabla)(\vec{V}_0,\nabla) - \\ \frac{1}{c^2}\left(\frac{\partial V_0}{\partial t},\nabla\right)\varphi = 0\end{aligned}\right\} \tag{3.7'}$$

This equation may be considered as the equation for the propagation of sound in a medium moving with velocity $\vec{V}_0(t)$, depending on the time but not depending on the coordinates. In fact, it almost agrees with the previously (Chapter I, section 5) derived equation (1.85) governing the propagation of sound in a medium in which the wind blows with constant velocity $\vec{V}_0$. The difference lies only in the presence of the last term containing the acceleration $d\vec{V}_0/dt$. If it is assumed, however, that the velocity of the wind $\vec{V}_0$ is a function of the time, an equation accurately agreeing with equation (3.7') would be obtained in section 5. The assumption of the presence of such wind is, of course, an artificial one, but it is compatible with the equations of the hydrodynamics of an incompressible fluid. These equations, in the presence of external volume forces $\rho\vec{g}$, are

$$\frac{\partial\vec{V}}{\partial t} + (\vec{V},\nabla)\vec{V} = -\frac{\nabla p}{\rho} + g; \ \mathrm{div}\ \vec{V} = 0 \tag{3.8}$$

With the assumption that $\vec{V}$ and p do not depend on the coordinates, there is obtained

$$\frac{d\vec{V}_0}{dt} = \vec{g} \tag{3.9}$$

It follows that such motion is realized in a fictitious field of gravity having an acceleration $\vec{g} = d\vec{V}_0/dt$. Thus, in considering the sound field of a moving source, the source is assumed as stationary; but it is then necessary, in general, to assume that the acceleration of a variable wind is conditioned by the "force of gravity" producing the acceleration

$$\vec{g} = -\frac{d\vec{v}}{dt} \tag{3.10}$$

15. Sound Source Moving Uniformly With Subsonic Velocity

An arbitrary sound source moving with constant velocity $\vec{v}$ less than the velocity of sound c will be considered. The velocity $\vec{v}$ is directed along the x-axis. Changing to a system of coordinates fixed to the sound source

$$\left.\begin{aligned} \xi &= x - vt \qquad & \eta &= y \\ \zeta &= z & \tau &= t \end{aligned}\right\} \tag{3.11}$$

yields a particular case of equation (3.7):

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial\tau^2} + \frac{2v}{c^2}\frac{\partial^2\varphi}{\partial\tau\partial\xi} - \frac{v^2}{c^2}\frac{\partial^2\varphi}{\partial\xi^2} = 0 \tag{3.12}$$

and introducing, as was done in section 5, a system of coordinates contracted along the x-axis

$$\left.\begin{aligned} \xi^* &= \frac{x - vt}{\sqrt{1 - \beta^2}} \qquad & \eta &= y \\ \zeta &= z & \tau &= t \end{aligned}\right\} \tag{3.13}$$

yields, in place of equation (3.12),

$$\left.\begin{aligned} \Delta^*\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial\tau^2} + \frac{2\beta}{\sqrt{1 - \beta^2}}\frac{1}{c}\frac{\partial^2\varphi}{\partial\tau\,\partial\xi^*} &= 0 \\ \Delta^* = \frac{\partial^2}{\partial\xi^{*2}} + \frac{\partial^2}{\partial\eta^2} + \frac{\partial^2}{\partial\zeta^2} & \\ \beta = \frac{v}{c} & \end{aligned}\right\} \tag{3.13'}$$

This equation agrees with equation (1.94),[12] and the generalized theorem of Kirchhoff (see section 6) may be applied to it. It is evidently sufficient to restrict this report to the consideration of the sound of frequency ω (in the system attached to the source), so that

$$\varphi = \psi e^{i\omega t} \tag{3.14}$$

On the basis of equation (1.108),

$$\left.\begin{aligned} \psi_P = \frac{1}{4\pi}\int\!\!\int_S \left\{\frac{\partial\psi}{\partial n} \cdot \frac{e^{-ikR}}{R^*} - \psi\frac{\partial}{\partial n}\left(\frac{e^{-ikR}}{R^*}\right)\right\} dS \; + \\ \frac{2i\beta k}{4\pi\sqrt{1 - \beta^2}}\int_S \varphi\,\frac{e^{-ikR}}{R^*} \cdot dS \end{aligned}\right\} \tag{3.15}$$

where ψ_P is the value of the potential at the point of observation P, and the surface S encloses the source. Further

$$R^* = \sqrt{\xi^* + \eta^2 + \zeta^2}; \quad R = \frac{-\beta\xi^* + R^*}{\sqrt{1 - \beta^2}} \tag{3.16}$$

where R^* signifies the distance (in the system ξ^*, η, ζ) from the point of ovservation P to the point of the surface $S(Q)$:

$$\left.\begin{aligned} \xi^* &= \xi_Q^* - \xi_P^* \\ \eta &= \eta_Q - \eta_P \\ \zeta &= \zeta_Q - \zeta_P \end{aligned}\right\} \tag{3.17}$$

The wave field far from the surface $S(R^* \to \infty)$ is now considered. For large distances from the point P from the surface, as is seen from figure 13,

$$R^* = R_P^* + R_Q^* \cdot \cos\theta_{PQ} + \cdots \tag{3.18}$$

where R^* is the distance OP, R_Q^* is the distance OQ, and θ_{PQ} is the angle between OP and OQ. On the basis of equations (3.18) and (3.17),

$$R = \frac{-\beta\xi^* + R^*}{\sqrt{1 - \beta^2}} = \frac{\beta\xi_P^* + R_P^*}{\sqrt{1 - \beta^2}} + \frac{-\beta\xi_Q^* + R_Q^* \cdot \cos\theta_{PQ}}{\sqrt{1 - \beta^2}} + \cdots = R_P + \Delta + \cdots \tag{3.19}$$

[12]It is necessary to bear in mind that β is now v/c, whereas in section 6, β denotes V_O/c; thus β in section 6 and here differ in sign (because $V_O = - v$).

where

$$\left.\begin{aligned} R_P &= \frac{\beta\xi_P^* + R_P^*}{\sqrt{1 - \beta^2}} \\ \Delta &= \frac{-\beta\xi_Q^* + R_Q^* \cos\theta_{PQ}}{\sqrt{1 - \beta^2}} \end{aligned}\right\} \tag{3.20}$$

Substituting the value of R (eq. (3.19)) in equation (3.15) and neglecting terms of the order $1/R^{*2}$ yields

$$\psi = \frac{e^{-ikR_P}}{4\pi R_P^*}\left\{\int_S\left(\frac{\partial\psi}{\partial n} - ik\psi\frac{\partial R_Q}{\partial n}\right)e^{-ik\Lambda}\cdot dS + \frac{2i\beta k}{\sqrt{1 - \beta^2}}\int\psi e^{-ik\Lambda}\, dS\right\} \tag{3.21}$$

The expression in braces depends only on the dimensions and form of the surface and the angles determining the direction of the radius vector OP. These angles are different depending on whether they are taken in the contracted system ξ^*, η, ζ or in the initial system ξ, η, ζ (they differ by a magnitude of the order of β^2). Let them be θ, χ in the system ξ, η, ζ (and θ^*, χ in the contracted system, respectively).

With the system ξ, η, ζ, the following may be written:

$$\psi(\xi_P,\eta_P,\zeta_P) = \frac{e^{-ikR_P}}{R_P^*}\cdot Q(\theta,\chi) \tag{3.22}$$

where ξ_P^* in R and R_P^* must be expressed in terms of ξ_P. On the basis of equation (3.14), the following is obtained for φ:

$$\varphi(\xi_P^*,\eta_P,\zeta_P,\tau) = \frac{e^{i\omega\left(t-\frac{R_P}{c}\right)}}{R_P^*}\cdot Q(\theta,\chi) \tag{3.23}$$

where $Q(\theta,\varphi)$ is the integral

$$4\pi Q(\theta,\chi) = \int_S\left(\frac{\partial\psi}{\partial n} - ik\psi\frac{\partial R_Q}{\partial n}\right)e^{-ik\Lambda}\cdot dS + \frac{2i\beta k}{\sqrt{1 - \beta^2}}\int_S\psi e^{-ik\Lambda}\cdot dS \tag{3.24}$$

The magnitude $Q(\theta,\chi)$ determines the force of the sound source (it has the dimensions of the volume velocity (cm^3/sec)) and its direction. If $Q(\theta,\varphi)$ is developed in a series of spherical functions $P_l^m(\cos\theta)e^{im\chi}$, where $l = 0,1,2,3,\cdots$, and $m = 0 \pm1, \pm2, \pm3,\cdots \pm l$, then

$$Q(\theta,\varphi) = \sum_{l=0}^{n}\sum_{m=-l}^{+l} Q_{lm}\cdot P_l^m(\cos\theta)\cdot e^{im\chi} \tag{3.25}$$

When all the coefficients Q_{lm}, except $Q_0 = Q$, are equal to zero, then a source of zero order results in

$$\varphi(\xi_P, \eta_P, \zeta_P, t) = \frac{e^{i\omega\left(t - \frac{R_P}{c}\right)}}{R_P^*} \cdot Q_0 \tag{3.26}$$

If, for example, only Q_{10} is different from zero, then, since $P_1^0 = \cos\theta$,

$$\varphi(\xi_P, \eta_P, \zeta_P, t) = \frac{e^{i\omega\left(t - \frac{R_P}{c}\right)}}{R_P^*} \cdot Q_{10} \cdot \cos\theta \tag{3.27}$$

that is, a dipole source where the dipole is oriented along the ξ-axis. The terms with $l > 1$ give multipole radiation.

Consideration will now be given to the dependence of φ on the distance. It is evident that the surfaces of constant amplitude ψ diverging in direction by angles included in $Q(\theta,\varphi)$ will be the surfaces

$$R_P^* = \text{constant} \tag{3.28}$$

But $R_P^* = \sqrt{\frac{\xi_P^2}{\sqrt{1 - \beta^2}} + \eta^2 + \zeta^2}$, that is, the surfaces of constant amplitude will be the ellipses (fig. 14)

$$\frac{\xi_P^2}{1 - \beta^2} + \eta^2 + \zeta^2 = \text{constant} \tag{3.29}$$

The surfaces of constant phase will be

$$\alpha = \omega\left(t - \frac{R_P}{c}\right) = \text{constant} \tag{3.30}$$

From this it is seen that the phase velocity along R_P is equal to the velocity of sound c. It is now assumed that the wave field φ is observed from the point of view of a stationary observer. On account of the motion of the sound source, R_P and, therefore, the wave phase α will then depend on the time t in a more complicated way than simple proportionality to t. Hence the observer will not perceive this sound field as a field of harmonic vibrations (although in the system attached

to the source harmonic vibrations were assumed). Nevertheless, if the changes in the magnitude R_P are not too rapid, the frequency ω' can be determined for the stationary observer as the derivative of the phase α with respect to the time

$$\omega' = \frac{d\alpha}{dt} = \omega\left(1 - \frac{1}{c}\frac{dR_P}{dt}\right) \tag{3.31}$$

The computation of the derivative dR_P/dt, on the basis of equations (3.20) and (3.18), yields

$$\frac{1}{c}\frac{dR_P}{dt} = \frac{\beta+\xi_P^*/R_P^*}{\sqrt{1 - \beta^2}} \cdot \frac{1}{c} \cdot \frac{d\xi_P^*}{dt} = - \beta \cdot \frac{\beta+\xi_P^*/R_P^*}{(1 - \beta^2)} \tag{3.32}$$

whence

$$\omega' = \omega \cdot \frac{1 + \beta \dfrac{\xi_P^*}{R_P^*}}{1 - \beta^2} \tag{3.33}$$

This formula gives an expression for the change of frequency caused by the motion of the sound source, that is, the Doppler effect produced by the motion of the source. If the observer is located ahead of the source, the following is obtained from equation (3.33):

$$\omega' = \frac{\omega}{1 - \beta} \qquad (\xi_P^* = R_P^*) \tag{3.31}$$

and, if behind the source,

$$\omega' = \frac{\omega}{1 + \beta} \qquad (\xi_P^* = - R_P) \tag{3.33'}$$

Equations (3.33) and (3.33') are the simplest formulas for the Doppler effect. Formula (3.33) gives the numerical expression of the Doppler effect for any position of the observer. If magnitudes of the order of β^2 are neglected, the following is obtained from formula (3.33):

$$\omega' = \omega(1 + \beta \cos \theta) \tag{3.34}$$

where θ is the angle between the velocity of the source and the direction OP toward the observer.

16. Sound Source Moving Arbitrarily but with Subsonic Velocity

The computation carried out in the preceding section shows that the field at a great distance from a uniformly moving source has the form of a field produced by a point source concentrated at the point O (see

fig. 13), and the nature of the source is entirely concealed in the function $Q(\theta,\varphi)$ determining the force and direction of the source. On the basis of this result the theorem of Kirchhoff may be avoided, which, although it can be formulated also for a nonuniformly moving surface, obtains in this case a form which is very complicated and unsuitable for applications. With the assumption that the source moves along the trajectory

$$\left.\begin{aligned} x &= X(t) \\ y &= Y(t) \\ z &= Z(t) \end{aligned}\right\} \tag{3.35}$$

The true nature of the source will be disregarded and the assumption will be made that the vibration is produced by a certain volume force concentrated at the location of the point source. The result will not depend on assumption (ref. 32). This assumption of the method of producing the vibrations is expressed by the fact that in the wave equation an expression determining the strength of the source is introduced on the right side:

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\ 4\pi Q(x,y,z,t) \tag{3.36}$$

In order to express the fact that the force Q is applied only at the location of the source, use is made of the δ functions introduced in section 6

$$Q(x,y,z,t) = F(t)\cdot\delta(x - X(t))\cdot\delta(y - Y(t))\cdot\delta(z - Z(t)) \tag{3.37}$$

The magnitude $F(t)$ gives the dependence of the force on the time in the system attached to the source. Due to the introduction of the δ functions, which are everywhere equal to zero except at the points where their argument becomes zero, the force Q will be different from zero only at the place where the source is located at the instant of time considered. The solution of equation (3.36) is evidently equivalent to the solution of equation (3.7) with a stationary right side:

$$Q(\xi,\eta,\zeta,\tau) = F(\tau)\cdot\delta(\xi)\cdot\delta(\eta)\cdot\delta(\zeta) \tag{3.37'}$$

that is, to the finding of a singular solution of equation (3.7'). The solution of the wave equation (3.26) with the right side present, as is known, reads (see section 6)

$$\varphi(x,y,z,t) = \int\frac{Q(x',y',z',t - r/c)}{r}\,dv' \tag{3.38}$$

where $r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}$ is the distance from the sound source (the point (x',y',z')) to the point of the observer (x,y,z). The evident physical sense of this solution consists in the fact that the disturbance formed at the point (x',y',z') does not at once reach the point (x,y,z) but is retarded by the time r/c; therefore the disturbance at the point (x,y,z) at the instant of time t is determined by the disturbance at the point (x',y',z') which was present at the instant of time $t - r/c$. Substituting now the value of equation (3.37) in equation (3.38) yields

$$\varphi(x,y,z,t) = \iiint \frac{[F]}{r} \delta(x' - [X]) \cdot \delta(y' - [Y])\delta(z' - [Z])\, dx'\, dy'\, dz' \tag{3.39}$$

where the brackets denote that the magnitude enclosed is taken at the time $t - r/c$. In order to carry out the integration, new variables which are arguments of the δ functions are introduced in place of x',y',z':

$$\left.\begin{aligned} A &= x' - [X] \\ B &= y' - [Y] \\ C &= z' - [Z] \end{aligned}\right\} \tag{3.40}$$

and dx',dy',dz' are transformed by the known formulas of integral calculus

$$dx'dy'dz' = \begin{vmatrix} \frac{\partial x'}{\partial A} & \frac{\partial y'}{\partial A} & \frac{\partial z'}{\partial A} \\ \frac{\partial x'}{\partial B} & \frac{\partial y'}{\partial B} & \frac{\partial z'}{\partial B} \\ \frac{\partial x'}{\partial C} & \frac{\partial y'}{\partial C} & \frac{\partial z'}{\partial C} \end{vmatrix} \cdot dA \cdot dB \cdot dC$$

$$= I \cdot dA \cdot dB \cdot dC \tag{3.41}$$

The determinant I is readily computed from formulas (3.40), and there is obtained

$$I = \left| \frac{1}{1 - \frac{\partial[X]}{\partial r} \cdot \frac{(x' - x)}{r} - \frac{\partial[Y]}{\partial r} \cdot \frac{(y' - y)}{r} - \frac{\partial[Z]}{\partial r} \frac{(z' - z)}{r}} \right|$$

$$= \frac{1}{\left|\left(1 - \frac{[V_R]}{c}\right)\right|} \tag{3.42}$$

where $[v_R]$ is the projection of the velocity of the source $\vec{v}$ in the direction of r taken at the instant of time $t - r/c$. The value of I is now substituted in equation (3.39) and the integration with respect to A, B, and C is carried out. On the basis of the properties of the δ functions, the result of the integration should simply be equal to the value of the function under the integral at the point $A = B = C = 0$ (see section 6), that is,

$$\varphi(x,y,t) = \sum \left(\frac{[F]}{r} \cdot I\right)_{A=B=C=0} \tag{3.43}$$

where the sum is taken over the points where $A = B = C = 0$. These points are easily determined. From the conditions $A = B = C = 0$ the following results:

$$\left.\begin{aligned} (x' - x) &= [X] - x \\ (y' - y) &= [Y] - y \\ (z' - z) &= [Z] - z \end{aligned}\right\} \tag{3.44}$$

By taking the square of these equations and combining term by term, an equation for obtaining the value of r at the point $A = B = C = 0$ is obtained. This value is denoted by R. By the method indicated the following equation results from equation (3.44):

$$R^2 = \left\{x - X\left(t - \frac{R}{c}\right)\right\}^2 + \left\{y - Y\left(t - \frac{R}{c}\right)\right\}^2 + \left\{z - Z\left(t - \frac{R}{c}\right)\right\}^2 \tag{3.45}$$

or

$$f(R) = 0 \tag{3.46}$$

where

$$f(R) = \left\{x - X\left(t - \frac{R}{c}\right)\right\}^2 + \left\{y - Y\left(t - \frac{R}{c}\right)\right\}^2 + \left\{z - Z\left(t - \frac{R}{c}\right)\right\}^2 - R^2 \tag{3.47}$$

Since $R > 0$, only the positive root of equation (3.46) is to be taken. On the basis of the equivalence of equations (3.44) and (3.46), the sum over the points $A = B = C = 0$ in equation (3.43) goes over into the sum over the positive roots of equation (3.46). The distance $r = R$ is the effective distance. Its physical meaning is illustrated by figure 15, where the trajectory of the source Q and the point of observation P are shown. If at the instant of time t the source is at point Q, the disturbance at the point P originates from the position Q', which it occupied at the instant $t - R/c$, where R is the distance $Q'P$; the instantaneous distance, however, $r = \sqrt{(x-X(t))^2 + (y-Y(t))^2 + (z-Z(t))^2}$ is equal to QP. Substituting in equation (3.33) the value $r = R$ yields

$$\varphi(x,y,z,t) = \sum \frac{F(t-R/C)}{R^{*}\sqrt{1 - \beta^{2}}} \tag{3.48}$$

where, as is easily verified by equations (3.42) and (3.47),

$$\sqrt{1 - \beta^{2}} \cdot R^{*} = R\left|1 - \frac{[v_R]}{c}\right| = \frac{1}{2}\left|\frac{df}{dR}\right| \tag{3.49}$$

If the velocity of the source is less than that of sound, there will be only a single positive root of equation (3.46). In fact, in order that the equation $f(R) = 0$ have a second positive root, $f(R)$ must pass through an extreme value, that is, df/dR must become zero. From equation (3.49) it is seen that in this case $[v_R]$ must be equal to c, which is impossible. Hence, for $v < c$,

$$\varphi(x,y,z,t) = \frac{F\left(t - \frac{R}{c}\right)}{R^{*}\sqrt{1 - \beta^{2}}} \tag{3.50}$$

where R is the only positive root of equation (3.46).[13] The case $v > c$ will be considered separately (section 20). From equation (3.50) it is seen that the wave field for all motions of the point source is expressed only through R^{*} and R, but the functions $R^{*}(x,y,z,t)$ and $R(x,y,z,t)$, since they are obtained from equation (3.46), are, of course, different. In particular for a uniform motion with velocity v along the x-axis

$$f(R) = \left\{x - v\left(t - \frac{R}{c}\right)\right\}^{2} + y^{2} + z^{2} - R_1 \tag{3.51}$$

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[13]In section 5 the solution has the form $F(t + R/c)/R^{*}$. The difference between them and equation (3.40) is only an apparent one. In the first place, the factor $\sqrt{1 - \beta^{2}}$ did not enter for the reason that in section 5 there was no interest in the absolute strength of the source. Further, equation (3.31) has also a formal leading solution. Thus, in equation (3.40), $Q(x',y',z',t + r/c)$ can be taken. The chosen sign + yields, in place of equation (3.40), $\varphi = F(t + R')/R^{*}\sqrt{1 - \beta^{2}}$, $R^{*} \cdot \sqrt{1 - \beta^{2}} = \left|1 + [v_R]'/c\right|$, where $[v_R]'$ is the value of v_R at the instant $t + R/c$. In equation (3.46) the sign before R would likewise change. The value of R would be R" (see fig. 15). From this it is seen that if equation (3.46) has the solution $R_1 = R$, it also has the solution $R_2 = - R''$. Hence, in order to obtain a lagging solution of equation (3.46), it is necessary to take $R > 0$ if starting from $Q(x',y',z',t - r/c)$ while it is necessary to take $R < 0$ if starting from $Q(x',y',z',t + r/c)$. But this root is precisely equal to $-R_1$.

From equation (3.46) the already familiar result is obtained

$$\left.\begin{aligned} R &= \frac{\beta\xi^* + R^*}{\sqrt{1 - \beta^2}} \\ R^* &= \sqrt{\xi^{*2} + y^2 + z^2} \\ \xi^* &= \frac{x - vt}{\sqrt{1 - \beta^2}} \end{aligned}\right\} \qquad (3.52)$$

The solution obtained (eq. (3.50)) represents the field of a zero source. By combining such sources, however, with the proper phases and disposing them according to a known method, a wave field having any directional characteristic can be represented. For example, two zero sources of the same strength but of oppostie phase placed at a small distance from each other ($l \ll R$) will give a dipole.

If the source began to function at a certain instant of time, for example, $t = 0$ (that is, if $F(t) = 0$ for $t < 0$), there would be present a wave front, that is, of a surface which would be reached by a disturbance starting out from the source.

From each position of the source a wave starts out at time t at the distance $R = ct$. Substituting this value of R in equation (3.46), the equation of the wave front is obtained:

$$\left\{x - X(0)\right\}^2 + \left\{y - Y(0)\right\}^2 + \left\{z - Z(0)\right\}^2 = c^2t^2 \qquad (3.53)$$

that is, a sphere of radius ct with center at the point where the source began to function (that is, at $x = X(0)$, $y = Y(0)$, $z = Z(0)$). Thus, for $v < c$, the moving source is at all times located within the sphere formed by the wave front (fig. 16).

The results obtained for the sound field of a moving source are, in many respects, in agreement with the known results of Lenard-Wichert for the electromagnetic field of a moving point charge (electron).

17. General Formula for Doppler Effect

If the source of sound is assumed harmonic and having in its own system the frequency ω, the form of φ (eq. (3.47)) is restricted:

$$\varphi(x,y,z,t) = \frac{Q \cdot e^{i\omega\left(t - \frac{R}{c}\right)}}{R^*\sqrt{1 - \beta^2}} = \frac{Q \cdot e^{i\alpha}}{R^*\sqrt{1 - \beta^2}} \qquad (3.54)$$

From the instantaneous frequency ω' perceived by a certain observer not moving together with the source, the derivative of the phase α with respect to the time is understood

$$\omega' = \frac{d\alpha}{dt} = \omega\left(1 - \frac{1}{c}\frac{dR}{dt}\right) \tag{3.55}$$

This formula must be considered as the most general formula for expressing the Doppler effect. It was presented earlier for uniform motion; it remains true also for the general case of motion. In section 15, however, the question of the limits of validity of this formula was not considered. For an observer not attached to the source, the spectrum of the wave field $\varphi(x,y,z,t)$, notwithstanding the harmonics of the source, will appear as continuous and the intensities of the individual frequencies will be determined by the amplitudes $\Psi(x,y,z,\omega)$ in the expression

$$\varphi(x,y,z,t) = \int_{-\infty}^{+\infty} \Psi(x,y,z,\omega')e^{i\omega't}d\omega' \tag{3.56}$$

It may be asked under what conditions the action of this entire frequency spectrum is equivalent to the action of a single one ω' which depends on the time according to equation (3.55). The answer to this question is simple and is connected with an analysis of the work of the sound receiver used by the observer. Let this receiver be a certain resonator with a time constant equal to T. In such a resonator the frequencies will be established in time T. If the time dependence of the force acting on the receiver is written in the form

$$\varphi(x,y,z,t) = \frac{Q}{R^*} \cdot e^{i\omega't} = Ae^{i\omega't} \tag{3.57}$$

where ω' is the "instantaneous" frequency (eq. (3.55)) and A is the "instantaneous" amplitude ($A = Q/R^*(t)$, the dependence of A and ω' on the time may be neglected under the conditions that

(1) A varies slowly by comparison with the changes of phase $\omega't$, that is,

$$\frac{1}{A}\cdot\left|\frac{dA}{dt}\right| \ll \omega' \tag{3.58}$$

(2) The frequency ω' changes little in the time T during which the frequencies are being established

$$\frac{d\omega'}{dt}\cdot T \ll \omega' \tag{3.59}$$

From the preceding it can be seen that the Doppler effect may be observed only for sources with sufficiently large damping (small T). These conditions will be analyzed in more detail; but now, if they are assumed satisfied, the Doppler effect will be considered for the case of an observer and a sound source moving uniformly and rectilinearly but at a certain angle to each other. On figure 17 is shown a source Q moving with velocity $\vec{v}$ and an observer P moving with velocity $\vec{V}$. The velocity of the observer relative to the source will be $\vec{u} = \vec{V} - \vec{v}$. In order to compute R, equation (3.30) is used. Substituting in R the value ξ^* and passing from motion along the x-axis to motion along any direction (which is done by simple rotation of the system of coordinates) yield

$$R = \frac{(\vec{r}, \vec{v}/c) + \sqrt{r^2(1 - v^2/c^2) + \left(\vec{r}, \frac{\vec{v}}{c}\right)^2}}{(1 - v^2/c^2)} \tag{3.60}$$

where $\vec{r}$ is the instantaneous distance $QP = \vec{r}_P - \vec{r}_Q$. Now, dR/dt can be computed, taking into account the fact that both the source and the observer are moving, so that

$$\left.\begin{aligned} \vec{r}_P &= \vec{V}t + \vec{r}_P^{\,0} \\ \vec{r}_Q &= \vec{v}t + \vec{r}_Q^{\,0} \end{aligned}\right\} \tag{3.61}$$

A somewhat long but simple computation leads to the following result for ω':

$$\omega' = \omega\left\{1 - \frac{\left(\frac{\vec{u}, \vec{v}}{c^2}\right)\left[1 + \frac{(\vec{v}, \vec{n})}{c}\right] + (1 - v^2/c^2)\,\frac{(\vec{n}, \vec{u})}{c}}{1 - v^2/c^2}\right\} \tag{3.62}$$

where the vector $\vec{n}$ is equal to

$$\vec{n} = \frac{\vec{r}}{\sqrt{r^2(1 - v^2/c^2) + \frac{(\vec{r}, \vec{v})^2}{c^2}}} \tag{3.63}$$

This is the most general formula for the Doppler effect for a uniformly moving source and observer. From this formula it is seen that, if they are relatively motionless ($\vec{u} = 0$), $\omega' = \omega$. For a motionless observer ($\vec{V} = 0$) there is obtained

$$\omega' = \omega \cdot \frac{1 + \frac{(\vec{n}, \vec{v})}{c}}{1 - v^2/c^2} \tag{3.62'}$$

and for a motionless source $(\vec{v} = 0)$

$$\omega' = \omega\left(1 - \frac{\vec{n}^0, \vec{V}}{c}\right); \quad \vec{n}^0 = \frac{\vec{r}}{r} \tag{3.62''}$$

For an estimate of $d\omega'/dt$ and dA/dt let $A \cong 1/r$. Condition (3.58) then reduces to

$$\frac{1}{r}\left|\frac{dr}{dt}\right| < \omega' \quad \text{or} \quad r \gg \frac{v}{\omega'} \tag{3.58'}$$

If the observation is made in the wave zone, then $r \gg 2\pi c/\omega'$; hence equation (3.58) is satisfied in all those conditions where, in general, the initial formulas derived for the wave zone are applicable. The case is otherwise with condition (3.59). If $d\omega'/dt = - d^2R/cdt^2$ is computed, with use of equations (3.60) and (3.61), then with an accuracy up to a magnitude of the first order with respect to v/c and V/c there is obtained

$$T \ll r\, c \frac{\left(1 - \frac{u_n}{c}\right)}{u_t^{\,2}} \tag{3.59'}$$

where u_n is the projection of the relative velocity on the direction of source to receiver and u_t is the projection on the direction perpendicular to this line. For a relative velocity u of the order of c for certain positions (small u_n), the magnitude of the time constant T should be much less than r/c and condition (3.59') may be very restrictive. When this condition is violated, the sound of the harmonic of the source itself will be received as an impulse containing different frequencies continuously distributed.

18. Sound of an Airplane Propeller

The sound of an airplane originates fundamentally from two sources: the propeller and the engine exhaust. The sound of the propeller likewise has a dual character. In the first place, a rotating body, such as the propeller of a motor, gives rise to periodic changes in pressure and velocity of the air near the plane and swept by it. These periodic changes of the air are accompanied by small compressions and rarefactions which are propagated in the form of a sound wave. The sounds of this origin are called rotational sounds.[14] In the second place, from the propeller blade, as from any moving body in the air, vortices are shed which likewise impart periodic impulses to the medium surrounding the propeller.

These periodic impulses are the cause of the second sound, the so-called vortical sound. In section 25 the origin of this sound and its

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[14]This term was introduced by E. Nepomnyashchii.

fundamental properties will be considered in detail. For the present, however, the discussion will be restricted to pointing out the fact that the frequencies of this sound are very high and are strongly absorbed in the air so that in observing the sound of a distant airplane only the rotational sound, and at that its lowest harmonics (and also the lowest harmonics of the exhaust), are heard. Hence, it will be entirely reasonable to consider in this section only the rotational sound. In figure 18 is shown the propeller of an airplane and its enclosing surface S on which the disturbances brought about by the motion of the propeller will be studied. The faces S' and S" of this surface (fig. 18) will be considered so far removed from the surface of rotation of the propeller that the motion of the gas on this surface may be assumed as linear (with the exception, of course, of the general forward motion of the air).

The possible frequencies of the rotational sound will be considered first. Let the propeller have n blades and make N rotations per second. It is then evident that at each point on the surface S, due to the rotation of the propeller, the state will be periodically repeated nN times per second so that the fundamental frequency (cyclic) of the rotational sound will be

$$\omega_0 = 2\pi nN \tag{3.64}$$

and its harmonic will be $\omega_m = \omega_0 m$, where

$$m = 2,3,4,\ldots$$

The computation of the intensity of the sound and its direction characteristic for these frequencies for a given shape of propeller and for a given speed presents exceptional difficulties.[15] Hereinafter the discussion will be limited to the investigation of the most general features of this sound and to qualitative estimates.

After the control surface S is shifted to the region where the periodic disturbances have become linear, the properties of the potential and its derivatives on the faces S' and S" of the surface S will be considered. A cylindrical system ξ^*, ρ, χ, rigidly attached to the airplane so that $\varphi = \varphi(\xi^*,\rho,\chi,t)$, will be taken as a system of coordinates.

Since the propeller rotates uniformly in the same plane in which the angle χ is measured, χ and t should enter φ only in the combination $t - \chi/\omega$, where $\omega = 2\pi N = \omega_0/n$ is the angular velocity of rotation of the propeller.

[15] See note, p. 93.

Expanding φ in a Fourier series with respect to the time t with period $T = 2\pi/\omega_0$ yields:

$$\left.\begin{aligned}\varphi(\xi^*,\rho,\chi,t) &= \sum_{m=-\infty}^{+\infty} \psi_m(\xi^*,\rho) \cdot e^{-im(\omega_0 t - n\chi)} \\ &= \sum_{m=-\infty}^{+\infty} \psi(\xi^*,\rho) \cdot e^{-im\cdot n\chi + i\omega_m t}\end{aligned}\right\} \tag{3.65}$$

In the following it is sufficient to consider separately each of the harmonics

$$\varphi_m = \psi_m(\xi^*,\rho)e^{i(\omega_m t - m\cdot n\cdot\chi)} \tag{3.66}$$

The theorem of Kirchhoff (section 6) is now applied to the potential of any of these harmonics and the wave field φ at a point P is considered at some distance from the airplane. According to equation (3.33),

$$\varphi_m(\xi_P^*,\eta_P,\zeta_P,t) = \frac{e^{i(\omega_m t - k_m R_P)}}{R_P^*} \cdot Q_m \tag{3.67}$$

where ξ_P^*, η_P, ζ_P are the coordinates of the point of observation P, and Q_m, on the basis of equation (3.24), is equal to

$$\left.\begin{aligned}4\pi Q_m = &\int_S \left(\frac{\partial\varphi_m}{\partial n} - ik_m \frac{\partial R_Q}{\partial n}\psi_m\right) e^{-ik_m\cdot\Delta} \cdot dS \;+ \\ &\frac{2i\beta k_m}{\sqrt{1-\beta^2}} \int \psi_m e^{-ik_m\Delta} \cdot dS_x\end{aligned}\right\} \tag{3.68}$$

where $\beta = v/c$, v is the velocity of the airplane, $k_m = \omega_m/c$, and, according to equation (3.20), the magnitude Δ is

$$\Delta = \frac{-\beta\xi_Q^* + R_Q^* \cdot \cos\theta_{PQ}}{\sqrt{1-\beta^2}} \tag{3.69}$$

The symbol Q is a point on the surface S (fig. 18). From the same figure it follows that

$$\cos\theta_{PQ} = \cos\theta_P^* \cdot \cos_Q^* + \cos(\varphi_P - \varphi_Q)\sin\theta_P^* \cdot \sin\theta_Q^* \tag{3.70}$$

$$R_Q^* = \sqrt{\rho^2 + h^{*2}} \qquad \xi_Q^* = h^* \tag{3.70'}$$

where h^* is the distance to the control surface, $\rho = \sqrt{\eta^2 + \zeta^2}$ is the distance from the axis of the propeller, θ_P^* and φ_P are the angles in the polar system determining the position of the point of observation P, and θ_Q^* and φ_Q are the same angles for the point Q of the surface S' (or S"). It is evident that $\cos\theta_Q^* = h^*/\sqrt{\rho^2 + h^{*2}}$ and $\sin\theta_Q^* = \rho/\sqrt{\rho^2 + h^{*2}}$. Substituting this value of Δ in equation (3.68) and ψ_m from equation (3.66), the integration with respect to φ_Q can be carried out. It is here necessary to bear in mind that

$$\int_0^{2\pi} e^{iz\cos(\chi-\chi')-im\cdot n\cdot\chi'}\,d\chi' = 2\pi i e^{-im\cdot n\chi}\cdot I_{mn}(z) \qquad (3.71)$$

where $I_{mn}(z)$ is a Bessel function of the first order $(m \cdot n)$. With use of equation (3.71) the following is obtained from equation (3.68) for the surface $S'(\xi_Q^* = h_1^*)$:

$$\left.\begin{aligned}4\pi Q_m' = 2\pi i e^{-im\cdot n\chi_P}\int_0^{r_0}\rho\,d\rho\cdot I_{m\cdot n}\left(\frac{k_m\,\rho\cdot\sin\theta_P^*}{\sqrt{1-\beta^2}}\right)\times \\ e^{i\cdot\frac{k_m h_1^*}{\sqrt{1-\beta^2}}\cdot(\beta-\cos\theta_P^*)}\left\{\left(\frac{\partial\psi_m}{\partial\xi^*}\right)_{\xi^*=h_1} - \right. \\ \left. ik_m\,\psi_m\cdot\cos\theta_P^* + \frac{2i\beta k_m}{\sqrt{1-\beta^2}}\,\psi_m^*\right\}\end{aligned}\right\} \qquad (3.72)$$

where r_0 is the radius of the control surface, which may be equated to the radius of the propeller, and the magnitude $\partial R_{QP}^*/\partial n = \partial R_{PQ}^*/\partial\xi$ is replaced for large R_{PQ} by $\cos\theta_P$. For the surface S_2 a similar expression Q_m'' is obtained which differs from equation (3.72) in the substitution of $-h_2^*$ for ξ^*. Combining Q_m' and Q_m'' yields

$$\left.\begin{aligned}Q_m = \frac{i}{2}e^{-im\cdot n\chi_P}\int_0^{r_0}\rho\cdot d\rho\cdot I_{m\cdot n}\left(\frac{k_m\cdot\rho\cdot\sin\theta_P^*}{\sqrt{1-\beta^2}}\right)\times \\ \left\{e^{i\chi_1}\cdot\left[\left(\frac{\partial\psi_m}{\partial\xi^*}\right)_1 + \left(\frac{2i\beta k_m}{\sqrt{1-\beta^2}} - ik_m\cdot\cos\theta_P^*\right)(\psi_m)_1\right] + \right. \\ \left. e^{i\chi_2}\left[-\left(\frac{\partial\psi_m}{\partial\xi^*}\right)_2 + \left(\frac{2i\beta k_m}{\sqrt{1-\beta^2}} + ik_m\cos\theta_P^*\right)\cdot(\psi_m)_2\right]\right\}\end{aligned}\right\} \qquad (3.73)$$

On account of the smallness of the magnitude $k_m h^*$, the phase multipliers $e^{i\chi_1}$ and $e^{i\chi_2}$ may be expanded in a power series in $k_m h^*$

$$\left.\begin{aligned} e^{i\chi_1} &= 1 + \frac{ik_m h_1^*}{\sqrt{1 - \beta^2}} (\beta - \cos \theta_P^*) + \cdots \\ e^{i\chi_2} &= 1 - \frac{ik_m h_2^*}{\sqrt{1 - \beta^2}} (\beta - \cos \theta_P^*) + \cdots \end{aligned}\right\} \tag{3.74}$$

The following is then obtained:

$$\left.\begin{aligned} Q_m = \frac{i}{2} e^{-im \cdot n\chi_P} \int_0^{r_0} \rho \, d\rho \times \\ I_{m \cdot n}\left(\frac{k_m \rho \sin \theta_P^*}{\sqrt{1 - \beta^2}}\right)\left[A_m(\rho) + B_m(\rho) \cos \theta_P^* + \right. \\ \left. C_m(\rho) \cos^2 \theta_P^* + \cdots\right] \end{aligned}\right\} \tag{3.75}$$

where $A_m(\rho)$ may be considered as the strength density of zero-order sources distributed in the plane of rotation of the propeller

$$\left.\begin{aligned} A_m(\rho) = \left(\frac{\partial \psi_m'}{\partial \xi^*}\right)_1 - \left(\frac{\partial \psi_m}{\partial \xi^*}\right)_2 + \frac{2i\beta k_m}{\sqrt{1 - \beta^2}} \cdot [(\psi_m)_1 - (\psi_m)_2] + \\ \frac{ik_m\beta}{\sqrt{1 - \beta^2}}\left[\left(\frac{\partial \psi_m}{\partial \xi^*}\right)_1 \cdot h_1^* + \left(\frac{\partial \psi_m}{\partial \xi^*}\right)_2 \cdot h_2^*\right] - \\ \frac{2\beta^2 k_m^2}{\sqrt{1 - \beta^2}} [(\psi_m)_1 h_1^* + (\psi_m)_2 h_2^*] \end{aligned}\right\} \tag{3.76}$$

the magnitude $B_m(\rho)$, as the strength density of dipole sources

$$\left.\begin{aligned} B_m(\rho) = - ik_m [(\psi_m)_1 - (\psi_m)_2] - \\ \frac{ik_m}{\sqrt{1 - \beta^2}}\left[\left(\frac{\partial \psi_m}{\partial \xi^*}\right)_1 h_1^* + \left(\frac{\partial \psi_m}{\partial \xi^*}\right)_2 \cdot h_2^*\right] \end{aligned}\right\} \tag{3.77}$$

the magnitude $C_m(\rho)$, as the density of quadripole sources

$$C_m(\rho) = - \frac{\beta^2 k_m^2}{\sqrt{1 - \beta^2}} \left[(\psi_m)_1 \; h_1^* + (\psi_m)_2 \cdot h_2^*\right] \tag{3.78}$$

and so forth. Therefore, the unknown functions ψ_m and $\partial\psi_m/\partial\xi^*$ are calculated now for $\xi^* = h_1^*$ and $\xi^* = - h_2^*$. These functions are independent of each other because the value of any one of them on the control surface S determines uniquely the solution of the wave equation. They can be given only in those cases where it may be assumed from some preliminary considerations that the assumed values of ψ_m and $\partial\psi_m/\partial\xi^*$ approximate the true values and are thus in agreement with each other. The computation of these magnitudes presents the fundamental problem for the computation of the sound of an airplane.[16] It is necessary to call attention to the following circumstance. In the integral (3.75), the magnitudes A_m, B_m, C_m,$\cdots$ must not change their signs as functions of ρ, at least in the region of most effective values of ρ (in the working part of the propeller blade). It is easily seen that the same refers also to the magnitude $I_{mn}(k_m\rho \sin \theta_P^*/\sqrt{1 - \beta^2})$. In fact, $k_m\rho = 2\pi nmN\rho/c = nmv(\rho)/c$, where $v(\rho)$ is the rotational speed of an arc of the propeller. The roots x'_{mn} of the equation $I_{mn}(x) = 0$ possess the property that $x'_{mn} > mn$, but $v/c < 1$. Hence, in the range of integration $0 < \rho < r_0$, the argument I_{mn} is less than x'_{mn}. Because of this I_{mn} can be moved outside the integral sign, replacing ρ by a certain effective value $\rho = R_0$. There is then obtained

$$\left.\begin{aligned} Q_m = \frac{i}{4\pi} e^{-im\cdot n\varphi_P} \cdot \pi R_0^2 I_{m.n}\left(\frac{k_m R_0 \sin \theta^*}{\sqrt{1 - \beta^2}}\right) \times \\ \left\{\vec{A}_m + \vec{B}_m \cdot \cos \theta_P^* + \vec{C}_m \cdot \cos^2 \theta_P^* + \cdots\right\} \end{aligned}\right\} \tag{3.79}$$

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[16]Attempts to compute these magnitudes have been frequently made (see the references at the end of the chapter, in particular, the book by E. Nepomnyashchii, "Investigation and computation of the sound of an airplane propeller"). These computations are not, however, entirely reliable because they make use of the relations of linear acoustics in the nonlinear region. In particular, no account is taken of the presence of a constant air flow; the magnitude $\rho\partial\psi/\partial t$ (where ρ is the air density) is equated to the pressure p on the blade of the propeller, whereas

$$p = \rho\partial\psi/\partial t - \rho \cdot (\nabla\psi)^2/2$$

and so forth. It is therefore difficult in this way to attain anything more than agreement in the most general features.

where $\vec{A}_m, \vec{B}_m, \vec{C}_m, \cdots$ are mean values of these magnitudes over the length of the propeller blade. Since the magnitudes $A_m, B_m, C_m, \cdots$ represent the coefficients of expansion in the small parameter $k_m h^*$, the value of $\vec{A}_m$ among the terms in braces in equation (3.79) should be predominant; that is, there is a source of zero order. Hence, the directional characteristic of the sound of the airplane propeller will be determined essentially by the factor I_{mn} while the remaining terms in equation (3.79) will only deform somewhat and displace the directional curve given by this factor. Since not only do the roots of the equation $I_{mn}(x) = 0$ exceed mn but also those values of x''_{mn} which correspond to the maximum $I_{mn}(x)$, the expression $I_{mn}(k_m\vec{R}_0 \sin \theta^*/\sqrt{1 - \beta^2})$ will monotonically increase with increase in θ^* to $\pi/2$ and then drop to 0 for $\theta^* = \pi$. Thus the maximum of the radiation will lie at $\theta^* = \pi/2$, that is, in a plane perpendicular to the line of flight of the airplane (in the plane of rotation of the propeller).[17]

This curve is given in figure 19 (curve a). In fact, there is generally observed an assymetry of the directional curve (curve b of fig. 19) which indicates that the part played by the dipole radiation can not be entirely ignored in comparison with the part played by the radiation of zero order. Both curves refer to a system of coordinates which are at rest relative to the airplane. Now the intensity in the sound spectrum of the propeller will be determined. For this the magnitude Q_m in equation (3.79) has the sense of a volume velocity. Its fundamental term contains the magnitude A_m equal approximately to the sum of the velocity components of the air normal to the surface S. These velocities are produced by the compression of the air in the motion of propeller blades and may be represented in direct dependence on the velocity of motion of these blades.

Consider the velocity component $u(t - \varphi/\omega, \rho, \xi^*)$ normal to the surface S_1. The same expansion (eq. (3.65)) in a Fourier series is applicable to it that applied for φ, namely,

$$u\left(t - \frac{\chi}{\omega}, \rho, \xi^*\right) = \sum_m u_m(\rho, \xi^*)\, e^{im(\omega_0 t - n\varphi)} \tag{3.80}$$

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[17]The difference between θ and θ_2^* is ignored since these angles differ by a magnitude of the order of β^2.

whence

$$u_m(\rho,\xi^*) \cdot e^{-inm\chi}$$

$$= \frac{1}{T}\int_0^{2\pi} u\left(t - \frac{\chi}{\omega},\rho,\xi^*\right) e^{-im\omega_0 t} \cdot dt \tag{3.81}$$

If the width of the blade at ρ is equal to $l(\rho)$, it may be assumed that u as a function of time has the form of an impulse lasting over the time $\tau = l/v = l/\omega\rho$, so that $u = u_0$ for $0 < l < \tau$ and $u = 0$ outside this interval. Carrying out the proposed integration in equation (3.81) yields

$$u_m(\rho,\xi^*) \cdot e^{-imn\chi}$$

$$= -\frac{u_0}{T} \cdot \frac{(e^{-im\omega_0 T} - 1)}{im\omega_0} = \frac{iu_0}{2\pi m}\left(e^{-im\frac{l}{\rho}} - 1\right) \tag{3.82}$$

From this it is seen that the amplitude of u_m very slowly decreases with increasing m so that the spectrum of the sound of the airplane should be very rich in harmonics, as is actually observed to be the case.[18]

19. Characteristics of Motion at Supersonic Velocity.

Density Jumps (Shock Waves)

Before the problem of immediate interest, that of sound radiation from a source moving with supersonic velocity, is discussed, consideration will be given to those special phenomena which arise in the flow about a body with velocity of motion exceeding the velocity of sound in the medium c.

The essential difference between a flow with $v > c$ and a flow with $v < c$ may be considered from the equation for the velocity potential Φ describing the flow of a compressible fluid. According to the generalized equation of Bernoulli (eq. (1.27')),

$$w = \int \frac{dp}{\rho} = \frac{\partial\Phi}{\partial t} - \frac{1}{2}(\nabla\Phi)^2 \tag{3.83}$$

[18]The assumptions herein were too simplified, of course, to expect anything more than a qualitative conclusion. The computation of the form of the impulses is carried out in the book by E. Nepomnyashchii. As previously pointed out, however, it would be necessary to choose values of the impulse on a suitable control surface, whereas generally their values are computed in the plane of the propeller.

On the other hand, the equation of continuity reads

$$\frac{\partial \rho}{\partial t} - \operatorname{div}(\rho \nabla \Phi) = 0 \tag{3.84}$$

(since $\vec{v} = -\nabla\Phi$). Noting that $\partial w/\partial t = \frac{1}{\rho}\frac{dp}{d\rho}\frac{\partial \rho}{\partial t} = \frac{c^2}{\rho}\frac{\partial \rho}{\partial t}$ and $\nabla w = \frac{c^2}{\rho}\nabla \rho$ and in equation (3.84) expressing $\partial\rho/\partial t$, $\nabla\rho$ in terms of $\partial w/\partial t$, ∇w, and w in terms of Φ with the aid of equation (3.83) yields

$$\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} - \Delta\Phi - \frac{1}{2c^2}\frac{\partial}{\partial t}(\nabla\Phi)^2 -$$

$$\left\{\nabla\Phi, \nabla\left[\frac{\partial \Phi}{\partial t} - \frac{1}{2}(\nabla\Phi)^2\right]\right\} = 0 \tag{3.85}$$

If a local system of coordinates x, y, and z is introduced such that the axis ox is directed along the normal to the surface Φ = constant (i.e., along the direction of the velocity $\vec{v}$ at the point considered) and the axes oy and oz lie in the tangent plane, equation (3.85) assumes the form

$$\left(1 - \frac{v^2}{c^2}\right)\cdot\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} + \frac{2v}{c^2}\cdot\frac{\partial^2\Phi}{\partial t\partial x} - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = 0 \tag{3.86}$$

If at a point of the flow the velocity v exceeds the local velocity of sound c, the coefficient before $\partial^2\Phi/\partial x^2$ becomes negative so that the coordinate x assumes, as it were, the same status as the time; the equation of elliptical type relative to the coordinates turns into an equation of the hyperbolic type. These two types of equations fundamentally differ from one another. The hyperbolic equation has discontinuous solutions which are not uniquely determined by the boundary conditions. A simple example illustrating this fact will subsequently be given. In fact, in the motion of a body at supersonic velocity, there arise in the medium the so-called density jumps or shock waves. These jumps are propagated over a great distance from the moving body along surfaces which for a small magnitude of the jump approximately coincide with the characteristics of equation (3.86). In the density jump the state of the medium changes discontinuously. Such discontinuous change is undergone simultaneously by all the magnitudes characteristic of the medium: the velocity, the density, the pressure, the temperature, and the entropy. By studying the propagation of the sound from a source moving with supersonic velocity, it would be systematic to start from that state of the medium which is produced by the motion of the source and to consider the sound as a small disturbance. However, at this time general methods of

solution of the problem of the supersonic flow about a body are not available; and, therefore, no theory is available which permits finding, in this case, the fields of velocity and pressure and determining the magnitude and position of the density jumps which arise with supersonic flows. For this reason the discussion will be restricted to the consideration of certain partial problems and to a qualitative analysis of the phenomena. Consideration will now be given to the simplest cases of supersonic motion which permit an uncomplicated mathematical analysis. The profiles of a thin infinitely long wing are shown in figure 20. The flow in this case is two dimensional and its velocity will be assumed as $v > c$. If it is assumed that the wing is thin (and the angle of attack small), the disturbance imparted by it to the flow may also be assumed small. Corresponding to this assumption, the potential $\Phi(x,y)$ is represented in the form

$$\Phi = - vx + \varphi(x,y) \tag{3.87}$$

where φ is a small correction and the higher powers of it may be neglected. Substituting equation (3.87) in equation (3.86) and neglecting terms containing higher powers and derivatives of φ yield

$$\left(1 - \frac{v^2}{c^2}\right) \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{3.88}$$

where c is the value of the velocity of sound in the undisturbed flow. Setting

$$x = \tau \cdot \sqrt{\beta^2 - 1} \qquad \beta = \frac{v}{c} > 1 \tag{3.89}$$

gives, in place of equation (3.79),

$$\frac{\partial^2 \varphi}{\partial y^2} - \frac{\partial^2 \varphi}{\partial \tau^2} = 0 \tag{3.90}$$

As also follows from the general theory, an equation of the hyperbolic type is obtained. If τ is considered as the time, it coincides with the equation for the propagation of waves in one dimension (y) with a velocity equal to 1.

The general solution of this equation has the form

$$\varphi = f_1 \ (\tau - y) + f_2 \ (\tau + y) \tag{3.91}$$

The disturbances, giving rise to φ, are disposed (along the wing profile) from $\tau = 0$ to $\tau = l/\sqrt{\beta^2 - 1}$ and are propagated according to equation (3.91) without change of their intensity along the lines $y = \tau$ and $y = -\tau$ (for example, PQ and P'Q' on fig. 20(a)). The assumption that $f_2 \neq 0$ for $y > 0$ would mean that the disturbance would travel ahead of the wing at any large distance. This contradicts causality and, therefore, it is assumed that $f_2 = 0$ for $y > 0$ and for the same reasons $f_1 = 0$ for $y < 0$.[19] Then

$$\varphi = f_1 (\tau - y) \qquad y > 0$$

and

$$\varphi = f_2 (\tau + y) \qquad y < 0 \tag{3.92}$$

With this choice of solutions the disturbances concentrate in the strips OABO' and OA'B'O'. The inclination of these strips is determined by the equation

$$y = \pm\tau = \pm \frac{x}{\sqrt{\beta^2 - 1}} \tag{3.93}$$

so that the angle ε = AOO', called the Mach angle, is equal to

$$\sin \varepsilon = \frac{1}{\beta} = \frac{c}{v} \tag{3.94}$$

The form of the functions f_1 and f_2 can now be connected with the form of the wing profile. Denoting the normal to the surface of the wing by $\vec{n}$, the following condition exists on the surface of the wing:

$$\frac{\partial \Phi}{\partial n} = - v \cdot \cos(x,n) + \frac{\partial \varphi}{\partial y} \cos(y,n) + \frac{\partial \varphi}{\partial x} \cos(x,n) = 0 \tag{3.95}$$

which expresses the fact that the components of the velocity normal to the wing surface are equal to zero. If the wing profile is thin and the angle of attack of the elements of its surface is everywhere small, $\cos(x,n) \cong 0$ and $\cos(y,n) \cong 1$. Hence the condition of equation (3.95) can be approximately written as

$$\pm \left(\frac{\partial \varphi}{\partial y}\right)_{y=0} = v \cos(x,n) \tag{3.96}$$

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[19] In this supplementary requirement there is also expressed the property referred to above of equations of the hyperbolic type.

The sign $+$ holds for the upper surface; the sign $-$ for the lower surface. Substituting φ from equation (3.92) yields for the upper surface

$$\left(\frac{df_1}{dy}\right)_{y=0} = -\frac{df_1(\tau)}{d\tau} = v \cdot \cos(x,n) \tag{3.97}$$

and since $\cos(x,n)$ is given on the wing profile as a function of x, and therefore also as a function of τ, there is thereby determined the potential $f_1(\tau)$ with an accuracy up to an unknown constant. In the same manner there is also found $f_2(\tau)$. From equations (3.92) and (3.97) an additional velocity on the x-axis is obtained

$$\Delta v_x = -\frac{\partial\varphi}{\partial x} = -\frac{df_1(\tau - y)}{d\tau\sqrt{\beta^2 - 1}} = \frac{v}{\sqrt{\beta^2 - 1}}\cos(x,n) \tag{3.98}$$

where $\cos(x,n)$ is considered as a function of $(\tau - y)$.

With the aid of equation (3.83) the change in pressure $\Delta p = p - p_0$ as compared with the pressure in the undisturbed flow p_0 can also be obtained. Thus, for small Δp, from equation (3.83)

$$\frac{\Delta p}{\rho_0} = \frac{\partial\Phi}{\partial t} - \frac{(\nabla\Phi)^2}{2} + \frac{v^2}{2} \tag{3.99}$$

The constant $v^2/2$ is so chosen that in the undisturbed flow, where $(\nabla\Phi)^2 \cong v^2$ and $\partial\Phi/\partial t = 0$, $p = p_0$. Substituting Φ from equation (3.77) and neglecting higher powers of φ and powers of the derivatives of φ yield

$$\Delta p = \rho_0 v \frac{\partial\varphi}{\partial x} \tag{3.100}$$

whence on the basis of equation (3.78)

$$\Delta p = \frac{\rho v^2}{\sqrt{\beta^2 - 1}} \cdot \cos(x,n) \tag{3.101}$$

At the point $x = 0$ (the point of meeting of the flow with the profile) $\cos(x,n) \geqslant 0$, and at the point of departure $(x = l)$ $\cos(x,n) \leqslant 0$. Outside the interval $0 < x < l$, $\cos(x,n) = 0$. Hence the pressure Δp and the velocity Δv have the form shown in figure 20. At the point of approach a discontinuity of the motion occurs. The resistance of a thin wing computed in this manner agrees well with test results (ref. 34). Both the pressure Δp and the velocity Δv_x maintain their values constant along the line $y = \pm\tau$, that is, along lines inclined to the flow by the Mach angle ε $(\sin \varepsilon = c/v)$.

The solution presented above, demonstrating the presence of discontinuities in the supersonic flow about a body, is suitable essentially for infinitely small density jumps. The theory of density jumps of finite magnitude can not be obtained from a consideration of only the differential equations of hydrodynamics since these equations lose their validity precisely in the region of discontinuity and must be replaced by suitable boundary conditions. In order to find these, a density jump of the form represented in figure 21 will be considered; equation (3.83) is the region of the undisturbed medium and equation (3.84) the region of the jump. Let the jump move with the velocity V in the positive direction of the x-axis. It is natural to take a system of coordinates in which the jump is at rest. In this system the velocities of the gas along the x-axis in regions of equations (3.83) and (3.84) will be

$$\begin{aligned} u_1 &= -V \\ u_2 &= U_2 - V \end{aligned} \tag{3.102}$$

where U_2 is the absolute velocity of the gas in the region of the jump. To derive the conditions at the jump it would be necessary to rewrite the fundamental equations of hydrodynamics in integral form. As was explained, however, in chapter I, these equations represent no other than the three laws of conservation and this fact may be utilized by applying these laws directly to the region of the density jump. The matter, momentum, and energy flows on both sides of the density jump must be the same. Making use of the expressions for these magnitudes (eqs. (1.9), (1.10), and (1.11)) and neglecting for the present the viscosity and the heat conductivity, the law of conservation of matter is obtained

$$\rho_1 u_1 = \rho_2 u_2 \tag{3.103}$$

where ρ_1 and ρ_2 are the density of the gas before and after the jump. Further, the law of the conservation of momentum is obtained

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \tag{3.104}$$

where p_1 and p_2 are the pressure before and after the jump, and finally the law of the conservation of energy is obtained

$$\frac{1}{2} \rho_1 u_1^3 + \rho_1 E_1 u_1 + p_1 u_1 = \frac{1}{2} \rho_2 u_2^3 + \rho_2 E_2 u_2 + p_2 u_2 \tag{3.105}$$

where E_1 and E_2 are the energy of unit mass of the gas before and after the jump.

With the use of equation (3.103), equation (3.105) may be written in the following form:

$$\frac{1}{2} u_1^2 + w_1 = \frac{1}{2} u_2^2 + w_2 \tag{3.105'}$$

where $w = E + p/\rho$ is the heat function. From these three equations are obtained

$$u_1 = -\sqrt{\frac{\rho_2}{\rho_1} \cdot \frac{p_2 - p_1}{\rho_2 - \rho_1}} = -V \tag{3.106}$$

$$u_2 = -\sqrt{\frac{\rho_1}{\rho_2} \frac{p_2 - p_1}{\rho_2 - \rho_1}} \tag{3.107}$$

and also with the use of the equation for an ideal gas

$$E = \frac{1}{\gamma - 1} \cdot \frac{p}{\rho} \qquad w = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} \tag{3.108}$$

the relation of Hugoniot[20] (ref. 35) is obtained

$$\frac{1}{(\gamma - 1)} \left(\frac{p_2}{\rho_2} - \frac{p_1}{\rho_1}\right) = \frac{1}{2} (p_1 + p_2) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) \tag{3.109}$$

Equations (3.106), (3.107), and (3.109) permit computing all the data referring to the density jump as soon as the pressure p_1 and the density of the gas ρ_1 ahead of the jump are given, and also one of the magnitudes characterizing the jump, for example, p_2.

In conclusion, the change in entropy occurring in a density jump will be computed. From equation (1.34) it follows that the entropy of unit mass of the gas is equal to

$$S = S_0 + c_v \ln \frac{p}{p_0} \left(\frac{\rho_0}{\rho}\right)^{\gamma} \tag{3.110}$$

From this equation the change in entropy is obtained

$$S_2 - S_1 = c_v \ln \frac{p_2}{p_1} \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} = c_v \ln \frac{p_2}{p_1} + c_p \ln \frac{\rho_2}{\rho_1} \tag{3.111}$$

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[20]This relation was earlier established by Rankin (ref. 36); see also reference 37.

If use is made of the relations of Hugoniot, it is not difficult to show that for a density jump this magnitude is greater than zero so that the processes in the jump have an irreversible character. It is precisely for this reason that it is impossible to restrict oneself to the differential equations of hydrodynamics which do not take into account such processes. The motion of the jump, as is seen, proceeds in the direction of increasing entropy since the gas has less entropy before the jump (eq. (3.87)) than after it (eq. (3.87)), and the jump moves in the direction from (2) to (1). The velocity of this motion $V = - u_1$ is readily found from the preceding equations if ρ_2 and ρ_1 are eliminated from equations (3.106) and (3.109). There is then obtained

$$u_1^2 = V^2 = \frac{c^2}{2\gamma} \cdot \left[(\gamma + 1) \frac{p_2}{p_1} + (\gamma - 1)\right] \tag{3.112}$$

where c is the adiabatic velocity of sound in the gas at rest (eq. (3.84)). Since $p_2 > p_1$, therefore $V^2 > c^2$; that is, the jump always moves with a velocity greater than the velocity of sound in the medium in which it originates. The relations herein derived will be used in analyzing the work of a sound receiver moving with a velocity greater than the velocity of sound in the medium.

20. Sound Source Moving with Supersonic Velocity and Having Small Head Resistance

In this section consideration will be given to the radiation of sound by a source moving with velocity $v > c$ and having a small head resistance. The theory of such a sound source is, to a considerable degree, analogous to the theory presented in section 19 of an infinitely thin wing. The sound source will be imagined as located on the body (fig. 22). The profile of the body will be given by a curve in a cylindrical system of coordinates (ρ,ξ,χ)

$$\rho = \rho_0(\xi) \tag{3.113}$$

The cross section of the body $\pi\rho_0^2$ will be considered infinitesimally small.

It is assumed further that the surface of this body or a part of it performs small vibrations of frequency ω. This vibration will be the sound source. The potential of the flow Φ will be given in the form

$$\Phi = - v \cdot x + \varphi_0 + \varphi \tag{3.114}$$

where $v \cdot x$ is the potential of the undisturbed flow, φ_0 is the potential produced by the motion of the body, and φ is the potential

produced by the vibrations of the surface of the body (it is proportional to $e^{i\omega t}$). The potential φ_0 is of no interest since in a system of coordinates connected with the body it does not depend on the time.[21] The assumption of the small cross section of the body permits restriction to the linear theory. In virtue of this the solution will be a superposition of the steady solution and the unsteady sound field. The problem thus reduces to the determination of Φ. For solving this problem the method of sources will be used. The field of a point source of sound moving with supersonic velocity will first be determined and then a suitable distribution of these sources over the surface of the body of revolution will be taken. In a system of coordinates attached to the body let there be a point source at the points ξ_0, η_0, ζ_0 lying on the surface of the body under consideration. In a stationary system of coordinates, the coordinates of this source will be

$$X = vt + \xi_0$$
$$Y = \eta_0$$
$$Z = \zeta_0 \tag{3.115}$$

The strength of this source dQ will be assumed as infinitesimally small and proportional to an element $d\sigma_0 = 2\pi\rho_0 \cdot d\xi_0$ of the surface of the body on which it is located

$$dQ = q(t,\xi_0,\eta_0,\zeta_0) \cdot d\sigma_0 \cdot \delta(x - vt - \xi_0) \times$$
$$\delta(y - \eta_0)(z - \zeta_0) \tag{3.116}$$

In this formula the small magnitude

$$dF = q(t,\xi_0,\eta_0,\zeta_0)d\sigma_0 \tag{3.117}$$

has the same meaning as F in equation (3.37). In correspondence with equation (3.43), the solution of the point source will be written in the form

$$\chi_0(x,y,z,t) = \sum \frac{[q]\,d\sigma_0}{R^*\sqrt{\beta^2 - 1}} \tag{3.118}$$

where $[q] = q(t - R/c,\xi_0,\eta_0,\zeta_0)$ and R and R^* are as previously determined from equations (3.46) and (3.49). However, in the case $v > c$

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[21]The potential φ_0 may be determined by a method similar to that presented in section 19 for a thin wing. See T. Kármán, reference 34, page 81.

the previous assertion on the uniqueness of the positive root of equation (3.46) is not true. Solving equation (3.46)

$$f(R) \equiv \left\{x - v\left(t - \frac{R}{c}\right) - \xi_0\right\}^2 +$$

$$(y - \eta_0)^2 + (z - \zeta_0)^2 - R^2 = 0 \tag{3.119}$$

yields

$$R = \frac{\pm R^* - \beta\xi^*}{\sqrt{\beta^2 - 1}} \qquad R^* = \sqrt{\xi^{*2} - \rho^2} \tag{3.120}$$

where

$$\xi^* = \frac{x - vt - \xi_0}{\sqrt{\beta^2 - 1}} = \frac{\xi}{\sqrt{\beta^2 - 1}}$$

$$\eta = y - \eta_0$$

$$\zeta = z - \zeta_0 \tag{3.121}$$

where, as will soon be shown, both roots of equation (3.120) are greater than zero. From expression (3.120) for R^* it is seen that ξ^{*2} must be greater than ρ^2 so that the entire solution lies within the cone

$$\xi^{*2} - \rho^2 \geqslant 0$$

that is,

$$\frac{\xi^2}{\beta^2 - 1} = \rho^2 \tag{3.122}$$

The generators of this cone start from the point $vt + \xi_0, \eta_0, \zeta_0$, at which the source is located, and, as is seen from equation (3.122), are inclined to the velocity v (to the axis ξ) by the Mach angle ε

$$\sin \varepsilon = \frac{c}{v} \tag{3.123}$$

With the possibility of a disturbance ahead of the excluded source, a restriction to the region $\varepsilon < 0$ (fig. 22) is necessary.[22] But $-\beta\xi^*$ for $\xi^* < 0$ is always greater than R^*. Therefore, R is positive and both solutions (eq. (3.120)) are lagging ones. The physical meaning of this double solution lies in the fact that at each point P (fig. 23) enclosed within the Mach cone two sounds arrive. If at the instant considered the source occupies the position Q, then Q' and Q" are two

[22]A similar assumption was made in the theory of a thin wing when f_2 for $y > 0$ and f_1 for $y < 0$ were neglected.

effective positions of the source from which the sound arrives at the point P at the instant t. At subsonic velocity there is only one effective position.

The solution for a point source does not have significance in the immediate neighborhood of the source (where it becomes infinite). From the computations it is seen that at supersonic velocity of the source there exists not only a singular point but an entire surface (the Mach cone) at which the solution becomes infinite. It follows that with restriction to a point source, it is impossible to assign a meaning to the solution (eq. (3.118)) not only near the source itself but also near the Mach cone. However, use may be made of this solution for constructing the field of a distributed source and also for a qualitative analysis of the phenomena for supersonic velocities. Assuming that q depends harmonically on the time t,

$$q = q_0(\xi_0,\eta_0,\zeta_0)e^{i\omega t} \tag{3.124}$$

a solution representing the field of an element of surface of the body is obtained from equation (3.118)

$$d\varphi = \frac{q_0 d\sigma_0}{R^*\sqrt{\beta^2 - 1}} \cdot \left[e^{i\omega\left(t-\frac{R_1}{c}\right)} + e^{ti\omega\left(t-\frac{R_2}{c}\right)}\right] \tag{3.125}$$

where R_1 and R_2 are the two roots of equation (3.119). This solution is valid within the Mach cone having its vertex at the point ξ_0,η_0,ζ_0. The total field due to all the elements of the surface carrying out a vibration with frequency ω will be

$$\varphi = \int \frac{q_0 d\sigma_0}{R^*\sqrt{\beta^2 - 1}} \cdot \left[e^{i\omega\left(t-\frac{R_1}{c}\right)} + e^{i\omega\left(t-\frac{R_2}{c}\right)}\right] \tag{3.126}$$

where the integral is extended over the region

$$\frac{\xi^2}{\beta^2 - 1} \geqslant \rho^2$$

$$\xi = x - vt - \xi_0 < 0$$

$$\rho^2 = (y - \eta_0)^2 + (z - \zeta_0)^2 \tag{3.127}$$

It is assumed that the radiating elements are disposed along rings from $\xi_0 = -l$ to $\xi_0 = 0$ so that in the cylindrical system of coordinates

ξ, ρ, χ q_0 depend on φ. Bearing in mind that $d\sigma_0 = \rho_0 \cdot d\chi \cdot d\xi$ and setting $q_0\rho_0 = a_0(\xi_0)/2\pi$, for $\rho_0 \to 0$, yield from equation (3.116) yields

$$\varphi = \int \frac{u_0(\xi_0)d\xi_0}{R^*\sqrt{\beta^2 - 1}} \left[e^{i\omega\left(t-\frac{R_1}{c}\right)} + e^{i\omega\left(t-\frac{R_2}{c}\right)}\right] \tag{3.128}$$

If the length of the wave is much greater than the dimensions of the radiator ($kl = \omega l/c \ll 1$), the phase factors may be taken outside the integral sign and this yields

$$\varphi = \frac{1}{\sqrt{\beta^2 - 1}} \left[e^{i\omega\left(t-\frac{R_1}{c}\right)} + e^{i\omega\left(t-\frac{R_2}{c}\right)}\right] \int \frac{a_0(\xi_0)d\xi_0}{R^*\sqrt{\beta^2 - 1}} \tag{3.129}$$

The last integral agrees with the integral considered by Kármán in his theory of the resistance of thin bodies at supersonic velocity and has everywhere a finite value (ref. 34). It may be noted that at a distance from the Mach cone where $R^* \gg l$, the magnitude R^* may be taken outside the integral sign and there is then obtained a quite simple result

$$\varphi = \frac{A_0}{R^*\sqrt{\beta^2 - 1}} \left[e^{i\omega\left(t-\frac{R_1}{c}\right)} + e^{i\omega\left(t-\frac{R_2}{c}\right)}\right] \tag{3.130}$$

where

$$A_0 = \int_{-l}^{0} a_0(\xi_0)d\xi_0 \tag{3.131}$$

The magnitude $a_0(\xi_0)$ must be determined from the condition that the derivative $-\partial\varphi/\partial\rho$ for $\rho \to 0$ should be equal to the velocity of a surface element carrying out vibrations with frequency ω. The method for determining $a_0(\xi_0)$ was given by Kármán in the preceding mentioned theory of the resistance of a thin body of revolution (ref. 34).

From the solution of equation (3.130), it follows that surfaces of constant amplitude will be the hyperboloids $R^* =$ constant, that is,

$$\frac{\xi^2}{\beta^2 - 1} - \rho^2 = \text{constant} > 0 \tag{3.132}$$

These hyperboloids are represented in figure 24. They asymptotically touch the Mach cone. For subsonic velocity the surfaces of constant amplitude are ellipsoids (see fig. 14) and for a stationary source they

are spheres. At a point of space P', lying outside the Mach cone, there will in general be no sound field and at each point P, lying within this cone, there will be two fields originating from the two effective positions of the sound source Q' and Q". With the assumption that the conditions at which a Doppler effect occurs (see section 17) are satisfied, the conclusion is drawn that at the point P there will be received two 'instantaneous' frequencies simultaneously

$$\omega' = \omega\left(1 - \frac{1}{c}\frac{dR_1}{dt}\right)$$

$$\omega'' = \omega\left(1 - \frac{1}{c}\frac{dR_2}{dt}\right) \tag{3.133}$$

In this case there thus occurs what might more properly be called not a Doppler displacement of the frequency but a Doppler splitting. The frequencies ω' and ω'' are easily computed on the basis of formulas for R_1 and R_2 (eq. (3.20))

$$\omega' = \omega\frac{\beta\dfrac{\xi^*}{R^*} - 1}{\beta^2 - 1}$$

$$\omega'' = \omega\frac{\beta\dfrac{\xi^*}{R^*} + 1}{\beta^2 - 1} \tag{3.134}$$

In particular, on the x-axis $(\rho = 0)$ is obtained

$$\omega' = -\frac{\omega}{\beta - 1}$$

$$\omega'' = -\frac{\omega}{\beta + 1} \tag{3.135}$$

From this it is seen that if $1 < \beta < 2$, then $|\omega'| > \omega$ and $|\omega''| < \omega$; but if $\beta > 2$, then both frequencies are less than ω, that is, in this case lowered tones (as compared with the initial (ω)) are heard.

21. Sound Field of a Sound Source for Supersonic Velocity of Motion

In the preceding section a sound source of infinitely small cross section moving uniformly with supersonic velocity was considered. With the assumption of a source of this shape, the entire problem was considered linearly; the state of the medium in this extreme idealized case was represented as a simple superposition of states, one of which

was determined by the motion of the body (solution of Kármán) and the other by the vibrations of its surface (radiation of sound). For finite dimensions of the cross section of the body, such simple superposition does not take place. The translational motion of a body of finite section produces in the medium considerable changes in the density, pressure and temperature and leads to the formation of density jumps (shock waves) of finite magnitude.

Because of the difference in the compression of the stream about a body at various points of the body, the velocity of sound c is not the same at all points. As a result the Mach angle, too, $\varepsilon = \arcsin c/v$, is different for different points of space about the body. The surfaces of discontinuity (of the density jumps) do not, for this reason, have the shape of a cone and only at a distance from the body do they possess this simple shape. In figures 25 and 26 are shown shock waves arising during the motion of artillery projectiles of various shapes obtained by the schlieren method.[23] Thus, the state of the medium near the body itself is very complicated, and, as has already been mentioned, the solutions of the hydrodynamic equations for this case are at present unknown.[24] At a great distance from the body, however, the situation is simpler. It may be imagined, at least for explaining the geometric and kinematic aspects of the matter, that the disturbance at some distance from the body is the result of the compounding of disturbances propagated with the velocity of sound from each point of the surface of the body. In this the differences in the velocity of sound propagation near the body must be unavoidably ignored, and therefore the assumed point of view essentially ignores the finite dimensions of the body so that a point source of sound would have to be considered. This, however, leads to an infinitely large magnitude of the shock wave on the Mach cone. Hence, the theory of a point source may be applied to the problem considered, restricting it, however, to a consideration of the kinematic side of the phenomenon. Such problems, for example, as the magnitude of the shock wave and its change with distance from the body cannot be considered from such a point of view. With these reservations the preceding theory (section 20) of the point source may be applied to the problem of the radiation of a sound source moving in an arbitrary manner. According to equation (3.48) the field of a harmonic point source is determined by the potential

$$\varphi = A_0 \sum_k \frac{e^{i\omega\left(t - \frac{R_k}{c}\right)}}{R_k^*} \tag{3.136}$$

[23]On photographing by the schlieren method see reference 38.

[24]Except for the case of the flow about a cone (see Ackeret, ref. 2).

where the magnitudes R_k (in the general case their number also may be greater than two) are determined as the positive roots of the equation

$$f(R) \equiv \left\{x - X\left(t - \frac{R}{c}\right)\right\}^2 + \left\{y - Y\left(t - \frac{R}{c}\right)\right\}^2 + \left\{z - Z\left(t - \frac{R}{c}\right)\right\}^2 - R^2 = 0 \tag{3.137}$$

and the magnitudes R_k^* are determined by the equation[25]

$$R_k^* = \frac{1}{2}\left|\frac{\partial f}{\partial R}\right|_{R=R_k} \tag{3.138}$$

It is evident that the surfaces

$$R_k^* = 0 \tag{3.139}$$

are no other than the surfaces representing the envelopes of the elementary disturbances propagated from the sound source. This makes possible a simple geometrical construction of the surfaces $R_k^* = 0$, which are the surfaces of the potential discontinuity, that is, of the shock-wave surfaces (strictly speaking, there is no justification in considering the state in the immediate neighborhood of these surfaces since on these surfaces $\varphi = \infty$).

For this purpose it is necessary to construct a family of spheres representing the fronts of the spherical waves issuing from the source at different instants of time and to draw the envelope of these spheres. In figure 27 this construction is shown for a uniformly moving sound source: (a) for subsonic velocity (in which case there is no envelope) and (b) for supersonic velocity. In this case the envelope is a Mach cone with vertex at the location of the source. From the latter construction there is clearly seen also the twofold character of the field for $v > c$: at the point P, at the instant of time assumed on the drawing, a disturbance arrives from the two points Q' and Q" (behind and ahead of P).

Density jumps are also frequently called shock waves or ballistic waves.

The suitability of these terms will be understood if the density jump is considered relative to a stationary observer or, in general, a sound (pressure) receiver. The density jump, moving together with the

[25]Here R_k^* differs from the preceding factor.

sound source, on passing by the sound receiver leads to a discontinuous increase of the pressure in the receiver with smoother pressure changes behind the jump. In figure 28 is shown the pressure in the ballistic B and nozzle wave N from a 305 millimeter shell according to a recording by E. Eksklagon. It is thus seen that the pressure in the receiver will have the character of an impulse or shock.

The problem of finding the envelope of the elementary disturbances may thus be considered as the problem of finding the front of the ballistic or shock wave. A rational analytical solution of this problem will also be presented. Equation (3.137) in the general case is transcendental with respect to R so that its direct solution may be very difficult. It is expedient to try to obtain the curves of the intersections of the wave-front surfaces with some coordinate plane in parametric form with R as a parameter. Equation (3.137) is quadratic with respect to x, y and z while equation (3.138) is always linear with respect to the same variables. By taking any section of the required surface with a plane, for example $z = z'$, y can be expressed by equation (3.138) as a function of x, z', and R; substituting in equation (3.137) yields a quadratic equation for x. Its roots will be

$$x = X_1(R,z')$$

$$x = X_2(R,z') \tag{3.140}$$

Substituting these values in equation (3.137) gives

$$y = Y_1(z',R)$$

$$y = Y_2(z',R) \tag{3.140'}$$

The equations $x = X_1$, $y = Y_1$, and $x = X_2$, $y = Y_2$ give the curve of intersection of the shock-wave front by the plane $z = z'$ in parametric form.

If the source originated at the instant of time $t = 0$, the values of the parameter R for the instant t lie in the interval

$$0 \leqslant R \leqslant c \cdot t$$

The instant of occurrence is the origin of a special disturbance source which on being propagated from the point of origin $(X(0), Y(0), Z(0))$ in the form of a spherical wave gives an additional sperical wave front which, generally speaking, is not a surface of discontinuity. At subsonic velocity of motion of the source, the wave front is entirely formed by this sphere having its center at the point of origin of the source (fig. 27(a)). An example of such a wave is that occurring at the instant of discharge of a missile from the barrel of a gun (discharge sound).

This wave is called a "muzzle wave" in contrast with the ballistic wave which moves together with the missile. Figure 29 shows a sketch of the muzzle and ballistic waves for a uniformly accelerated motion of a disturbance source starting at the point 0 at instant $t = 0$. From the diagrams shown for $t = 1, 3, 4$, and 8, the reader can follow the development of the ballistic wave which, in this example, overtakes the muzzle wave. The cross-hatched regions are those in which there are two effective positions of the sound source and at which, therefore, there should be heard the superposition of two sound fields, one starting from the side of the sound source, the other from the opposite side (see positions Q'', Q', Q, and P in fig. 23).

In figures 30 and 31 are shown the development of a ballistic wave of an artillery missile for a uniformly retarded (fig. 30) and uniformly accelerated (fig. 31) motion.[26] Figure 32 shows a ballistic (G) and a muzzle (M) wave for an accelerated curvilinear motion of a disturbance source.

[26]Figures 30, 31, and 32 were taken from the paper of L. Prandtl (ref. 39); figure 28 from the book by E. Eksklagon (ref. 40).

CHAPTER IV

EXCITATION OF SOUND BY A FLOW

22. General Data on Vortical Sound and Vortex Formation

The most common reason for the occurrence of sound in a medium is the periodic motion of bodies immersed in the medium and having a sufficiently high frequency, for example, the vibration of the prongs of a tuning fork, the rotational motion of the blades of an airplane or ship propeller, and so forth. The occurrence of sound is not restricted, however, to only such cases as these. Sound also arises when there is a constant flow about stationary solid bodies (or, what amounts to the same thing, in the motion of bodies with constant velocity) when it would appear that there were no causes that give rise to periodic phenomena. An example of this type of sound formation is provided by the whistle of the tension rods of airplanes, the rigging of ships, the sound of wires and strings ("aeolian harp"), and the whistling in the flow about angles, slots, and so on. It is important in this connection to note that the capacity of the string, for example, for vibrating plays a secondary part because these sounds occur also in the flow about nonyielding solid bodies. The initial causes for the occurrence of sound in these cases are not connected with the vibrations of bodies but with the phenomena of vortex formations in the flow of a fluid about bodies. The corresponding sound is therefore called a vortical sound. The twofold origin of the sound of an airplane propeller has already been pointed out. On the one hand, the sound of the propeller is caused by the periodic motion of the blades (rotational sound); and, on the other hand, a flow takes place about the propeller blades which leads to vortex formation and also to the occurrence of a particular vortical sound. The fundamental laws of vortex formation in the flow about bodies are now considered in more detail. However small the viscosity of the medium about the body may be, the very existence of the frictional forces produced by it leads to the formation of vortices in the initially potential flow. In order to clarify this aspect of the problem, use will be made of the equation of motion of a viscous, imcompressible fluid (the equation of Navier-Stokes). According to equation (1.13), setting $\operatorname{div} \vec{v} = 0$ gives

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + (\vec{v},\nabla)\vec{v}\right] = -\nabla p + \mu\Delta\vec{v} \qquad \operatorname{div} \vec{v} = 0 \tag{4.1}$$

The flow about a body of characteristic dimension d is now considered, and the velocity of the approaching flow is denoted by v. In place of x, y, z, and t, the nondimensional variables $x' = x/d$, $y' = y/d$,

$z' = z/d$, and $\tau' = tv/d$ are used; and $p = p'\rho v^2$ and $\vec{v} = v \cdot \vec{v}'$ are set up. Equation (4.1) then reduces to the nondimensional form

$$\frac{\partial \vec{v}'}{\partial t'} + (\vec{v}', \nabla') \vec{v}' = - \nabla' p' + \frac{1}{Re} \Delta' \vec{v}' \tag{4.1'}$$

where Re is the Reynolds number

$$Re = \frac{vd}{\upsilon} \tag{4.2}$$

($\upsilon = \mu/\rho$, the kinematic viscosity).

From the hydrodynamic equation thus reduced to the nondimensional form, it is seen that at large Reynolds numbers Re the last term in equation (4.1) may be rejected; and, therefore, in this case the viscous stresses play a vanishingly small part in comparison with the effects arising from the inertia of the fluid. The equations of motion of an ideal fluid are thus obtained. Hence, if the approaching flow were potential, it would have to remain so. This conclusion, however, is true only at a large distance from the body and is not true in the immediate neighborhood of and behind the body. The velocity $\vec{v}$ on the surface of the body itself is, because of the adherence of the fluid, equal to zero. Far from the surface it assumes a value close to that of the approaching flow ($v' = 1$). This change of velocity occurs in a thin layer which is called the boundary layer[27]. The thickness of this layer δ may be estimated from the fact that in this layer the action of the viscous stresses is comparable with the effect produced by the inertia. This means that in this layer the last term in equation (4.1) is comparable with the remaining terms. These latter terms are of the order of 1. Since in the boundary layer the velocity over its thickness δ varies from 0 to 1, the magnitude of the derivative $\partial^2 v/\partial n^2$, where n is the normal to the surface of the body, will be of the order of v/δ^2; and in nondimensional form the magnitude $\Delta v = \partial^2 v/\partial n^2 + \partial^2 v/\partial s^2$ (s is the tangential length) will be
$\Delta' v' = (l^2/\delta^2)\partial^2 v'/\partial n'^2 + \partial^2 v'/\partial s'^2 \cong l^2/\delta^2$ (instead of $\Delta' v' = l^2/l^2 = 1$ outside the boundary layer). From this it is concluded that in the boundary layer $1/Re \cdot l^2/\delta^2 \cong 1$; that is,

$$\delta = \frac{l}{\sqrt{Re}} \tag{4.3}$$

In this thin layer the flow may be considered as corresponding to the potential flow of an ideal fluid. The existence of a boundary layer, however thin it may be (large Re, small viscosity), leads to essential changes in the flow behind the body. In figure 33(a) is represented the potential flow about a cylinder and in figure 33(b) the flow as it is

[27]For details on the boundary layer and vorticity, see references 41 and 42.

actually obtained. The thin boundary layer b'b" becomes unstable at the point b" and gives rise to vortices shed from the body. The formation of these vortices may be explained in the following manner: The stream line a'a"a'" of the potential flow near the surface of the body is considered. In the region a'a" the stream moves with acceleration, and the pressure at a" drops, as follows directly from the law of Bernoulli

$$\frac{p}{\rho} + \frac{v^2}{2} = \text{constant} \tag{4.4}$$

because of the narrowing of the stream in the region a". On the other hand, in the region a"a'" the stream moves against an increasing pressure and is consequently retarded. In the ideal case of an absolutely nonviscous fluid, the particles of the fluid successfully overcome this rise in pressure, converting the stored-up kinetic energy into potential. In the presence of friction, however, part of the kinetic energy is spent in overcoming the forces of friction, and the store of kinetic energy of the particles is now insufficient for overcoming the increasing pressure. As a result, a reverse flow arises in this region. The point of occurrence of this flow b" is called the point of separation of the boundary layer. The picture of the flows that arise here is represented in more detail in figure 34. This reverse flow forms a vortex which gradually increases, approximately up to the dimensions of the body, and which finally breaks away from the body (fig. 33). The same also takes place at the lower point of separation. The development of the vortex on one side, however, hinders the development on the other. Hence, the development of the vortices and their separation occurs alternately, now one side, now on the other side of the body. The separating vortices form behind the body a double chain of vortices which are gradually dissipated. This double chain of vortices is termed a Kármán vortex street. The theory of this concept will be discussed subsequently. For the present, it is merely pointed out that so far no mathematical computation of the periodic separation of the vortices has been obtained. By numerical methods, Boltze (ref. 43) has succeeded in showing mathematically the development of a vortex behind the point of separation. Figures 35 and 36 are photographs of a developing vortex in the flow about a cylinder and also a Kármán street formed behind the cylinder at Re = 250. Although the frequency of the separation of vortices cannot as yet be computed mathematically, important conclusions can nevertheless be derived from dimensional considerations. From the magnitudes characterizing the flow about the body, v, the flow velocity; d, the dimension of the body; and υ, the kinematic viscosity, two magnitudes f and f' can be formed having the dimensions of frequency

$$f = \varkappa(\mathrm{Re}) \frac{v}{d} \tag{4.5}$$

$$f' = \varkappa'(\mathrm{Re}) \frac{\nu}{d^2} \tag{4.5'}$$

where $\varkappa$ and $\varkappa'$ are nondimensional coefficients depending on the Reynolds number. The first of the frequencies is a characteristic frequency of the possible periodic motions of the fluid at large values of the Reynolds number when the effect of the inertia of the fluid predominates; the second, on the contrary, is of significance in the case of predominant viscosity (small Re). Vortices arise only at large Reynolds numbers and therefore it may be expected that the frequency of separation of the vortices should be determined by equation (4.5). It may appear strange that the frequency of the vortices arising exclusively from the viscosity of the fluid is determined by equation (4.5) and not by equation (4.5'). This paradoxical character is, however, only an apparent one. If it is desired to make use of equation (4.5') for determining the frequency of separation of the vortices, then the magnitude d would have to denote not the dimension of the body but the thickness of the boundary layer δ. When δ is substituted from equation (4.3) into equation (4.5'), a result agreeing with equation (4.5) is obtained for f'. It may be remarked that equations (4.5) and (4.5') differ, of course, only in that $\varkappa$ and $\varkappa'$ in both cases depend little on Re. This phenomenon is, in fact, observed in actual cases. The periodic separation of the vortices with frequency (equation (4.5)) gives rise to periodic impulses of small compressions and rarefactions which are propagated at a distance from the body in the form of a sound wave the frequency of which agrees with f. This is the wave which is denoted as the vortical sound. The frequency of the vortical sound was first investigated by Strouhal for a vibrating string in an air flow (aeolian harp). From his tests, Strouhal derived precisely equation (4.5) with $\varkappa(\mathrm{Re}) = 0.185$.[28] The value of the Strouhal coefficient depends on the shape of the body, on the choice of the characteristic dimension d, and not much (in a certain interval of Reynolds numbers) on the Reynolds number. For a sphere or cylinder, d denotes the diameter. For a plate having width l and thickness b at angle of attack α to the flow, $d = l \sin \alpha + b \cos \alpha$.

For such determination of d, from test data (refs. 33 and 45)
For a cylinder: $\varkappa = 0.20$ in the range $10^3 \leq \mathrm{Re} \leq 3 \cdot 10^4$
For a plate: $\varkappa = 0.165$ to 0.180 for $10^3 < \mathrm{Re} < 1.8 \times 10^5$ (at angles of attack 20° to 90°).

The values of $\varkappa$ obtained by different authors differ little from one another. A more detailed investigation of the spectrum of vortical sound (ref. 45) shows that the equation of Strouhal (equation (4.5)) must be generalized in order to take into account the upper harmonics of the

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[28]This derivation of Strouhal was disputed on the basis that the investigated string is itself capable of vibrations. However, later investigations (see, for example, refs. 45 and 46) confirmed the equations of Strouhal for rigid bodies where the vibrations are due exclusively to the vortices.

fundamental frequency

$$f_n = \varkappa(\mathrm{Re})\,\frac{v}{d}\,n \qquad n = 1, 2, 3, \cdots \tag{4.6}$$

These harmonics, although weakly expressed, are nevertheless observed[29] (fig. 37)[30]. The intensity of the Strouhal sound has been investigated in considerably less detail. According to the observations of W. Holle (ref. 46) for the flow about thin cylinders (diameter d, length l), the intensity of the vortex sound at the distance r from the radiator is equal to

$$I = \alpha \cdot \frac{l d v^n}{r^2}, \mathrm{db} \tag{4.7}$$

where Holle assumes for n the value 7 (in general, according to his tests, $6 < n < 8$) and $\alpha = 5 \cdot 10^{-24}$ in centimeter-gram-second units. According to the observations of Yudin (ref. 47) on the intensity of the sound of a blower, $n = 6$ is obtained. The same result for n has been arrived at by Y. M. Sukharevskii from observations of the vortex sound in a wind tunnel (unpublished computation of the Physics Institute of the Soviet Academy of Sciences). Nepomnyashchii (ref. 33) from measurements on the vortical sound of a propeller (fig. 38) arrived at the value $n = 4$, which corresponds to the theoretical curve 40 log v (fig. 38). In fact, at least for large angles of attack ($\alpha > 20^{\circ}$), the curve 60 log v corresponding to $n = 6$ better corresponds with the results of his measurements. With regard to the directional characteristic of the vortex sound, the observations of Yudin (ref. 47) show that it agrees with the direction characteristic of a dipole the axis of which is perpendicular to the direction of flow about the body (e.g., a propeller radiates a vortical sound primarily in the direction of its axis symmetrical in front and behind). In figure 39 is shown the directional characteristic for the vortical sound of a propeller. The theoretical explanation of these laws will be given subsequently. For the present, the fact is noted that the high degree of dependence of the intensity of the vortex sound on the velocity of the flow ($n \geqslant 6$) has often appeared paradoxical because from the dimensions of the magnitudes it was assumed that, since the intensity of the sound is proportional to the square of the pressure and $\rho v^2/2$ is a measure of the pressure, n should be equal to 4. The error of this reasoning is based on the fact that the magnitude $\rho v^2/2$ is a measure of the pressure only in an

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[29]For large Reynolds numbers ($\mathrm{Re} > 10^5$), the expressed vortical frequency evidently does not, in general, exist. The spectrum of the vortical sound becomes practically continuous and the Strouhal frequency (equation (4.5)) becomes only a suitable measure for the frequencies represented in such a spectrum.

[30]Figure 37 is taken from the article by Holle (ref. 46).

incompressible fluid. In the wave region far from the body, this pressure, which decreases in inverse proportion to the square of the distance, is practically equal to zero; but the important part of the pressure, which decreases in inverse proportion to the first power of the distance, is entirely connected with the presence of compressibility of the gas or liquid. In general, from considerations of dimensionality, it can be concluded that n is greater than 4. In fact, the density of the flow of sound energy is equal to

$$I = \frac{\pi^2}{\rho c} \tag{4.8}$$

where π is the pressure in the wave, ρ the density of the medium, and c the velocity of sound.

If π is measured in units of $\rho v^2/2$, then

$$I = \chi \cdot \frac{\rho v^4}{c} \tag{4.9}$$

where the nondimensional coefficient χ may depend on the Reynolds number; the Mach number v/c; the ratio l/r, where l is some dimension of the body; and on the observation angles θ, φ. Since at large r, on account of the law of conservation of energy, I must be inversely proportional to the square of the distance, the following expression applies for $r \to \infty$:

$$\chi\left(\frac{l}{r}, \mathrm{Re}, \frac{v}{c}, \theta, \varphi\right) = \frac{l^2}{r^2} \cdot \chi'\left(\mathrm{Re}, \frac{v}{c}\, \theta, \varphi\right) \tag{4.10}$$

In place of l^2, it is possible, of course, to take the product ld. Further, in the absence of compressibility $(c \to \infty)$, $\chi' = 0$ (since the sound in the absence of compressibility is not radiated). Hence, χ' must be proportional to a certain positive power of the Mach number (v/c). In this way there is obtained

$$I = \alpha \cdot \frac{ld}{r^2} \cdot \frac{\rho v^4}{c} \cdot \left(\frac{v}{c}\right)^s \tag{4.11}$$

where α depends on Re and on the angles θ, ψ, and $s > 0$. The dependence of α on the Reynolds number in the range where the resistance of the body depends little on Re must be weak so that physically α depends only on the angles and determines the directional characteristics of the sound.

Additional considerations permit determining also the least value of s. The sound source may be assumed as a zero-order source (α does not depend on θ, φ), a dipole ($\alpha = \alpha' \cos^2\theta$, where the direction of the dipole axis cannot be indicated, and $\theta = 0$), and so forth. It will now be shown that the zero-order source should be excluded. In fact,

the strength of a zero-order source Q is equal to the volume velocity

$$Q = \int_{(S)} v_n dS \qquad (4.12)$$

where the integral is taken over the surface S about the source near the latter itself, and v_n is the normal component of the velocity of the fluid to the chosen surface S. In this region, the fluid may be considered as imcompressible since the wave length λ of the vortex sound

$$\lambda = \frac{c}{f} = \frac{1}{\varkappa} \frac{c}{v} d \qquad (4.13)$$

(for $v < c, \varkappa < 1$) is always much greater than the body. But for an incompressible fluid the flow through a closed surface is equal to zero. Hence, $Q = 0$.[31]

A diople radiation may thus be expected.[32] Since its strength is is also proportional to the velocity of flow v while the intensity of the radiation of a dipole source is proportional to the square of its strength (i.e., v^2) and to the fourth power of the frequency ($f^4 \sim v^4$), the result that the exponent s in equation (4.11) must be equal to 2 is obtained; that is,

$$I = \alpha'(Re) \cdot \cos^2\theta \cdot \frac{ld}{r^2} \cdot \frac{\rho v^6}{c^3} \qquad (4.11)$$

The magnitude of the coefficient α' cannot, of course, be determined from dimensional considerations. With regard to the direction of the dipole axis ($\theta = 0$), it should, at least for symmetrical bodies, be surmised, on the basis of the symmetrical succession of the separation of the vortices from the upper line of separation and from the lower, that the axis of the dipole is directed along a line perpendicular to the flow (see section 25).

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[31] A possible objection to this conclusion is that for the radiation of sound the compressibility is essential, and that taking it into account gives $Q \neq 0$. But taking the compressibility into account means expansion in powers of v^2/c^2; hence, Q would be proportional to $v \cdot v^2/c^2$ and the square of Q proportional to v^2v^4/c^4. Since the intensity of a zero-order source is further proportional to $f^2 \sim v^2$, there would be obtained $I \sim v^8$, i.e., the succeeding term after equation (4.11) in the expansion in powers of v/c.

[32] The same conclusion was arrived at by E. Y. Yudin on the basis of his measurements of the directional characteristics of the sound of blowers.

23. Theory of the Kármán Vortex Street - Computation of the Frequency of Vortex Formation

Kármán and Rubach (refs. 48 and 49) succeeded in constructing a theory of a double chain of vortices representing an idealization of the vortex street which actually arises behind bodies moving in a fluid (see fig. 36).

The Kármán-Rubach theory refers to the flow about infinite cylinders and plates so that only two-dimensional flow is considered in a plane (x,y) coinciding with the plane of the cross section of the body. Along an axis parallel to the generator of the cylinder or plate, the flow is assumed unchanged.

The two-dimensional flow of an ideal incompressible fluid may, as is known, be described by a complex velocity potential $\Phi(z)$, $z = x + iy$.[33] The components of the velocity $\vec{v}$ along the ox and oy axes, v_x and v_y, are computed from this potential by the formula

$$v_x - iv_y = -\frac{d\Phi}{dz} \tag{4.12}$$

The component along the third axis, however, in view of the assumption of the two-dimensional character of the flow, is equal to zero. If Φ is known, the pressure p can also be found. If Φ depends on the time, p is computed from the generalized Bernoulli equation

$$p = p_0 + \rho R\left(\frac{\partial\Phi}{\partial t}\right) - \frac{\rho}{2}\cdot\left|\frac{d\Phi}{dz}\right|^2 \qquad \left|\frac{d\Phi}{dz}\right|^2 = v_x^2 + v_y^2 = v^2 \tag{4.13}$$

In this expression, $R(\partial\Phi/\partial t)$ denotes the real part of $\partial\Phi/\partial t$. The complex potential of one vortex filament at the point $z = z_k$ will be

[33] If $\Phi(z,t) = \varphi(x,y,t) + i\psi(x,y,t)$ is an analytic function of z, then φ and ψ satisfy the equations of Laplace, $\Delta\varphi = 0$, and $\Delta\psi = 0$, where the derivatives of φ and ψ are subject to the Cauchy-Riemann conditions: $\partial\varphi/\partial x = \partial\psi/\partial y$, and $\partial\varphi/\partial y = -\partial\psi/\partial x$. If φ is taken as the velocity potential, the equation ψ = constant will give the streamlines orthogonal to the surfaces φ = constant. The velocities v_x, v_y are $-\partial\varphi/\partial x$ and $-\partial\varphi/\partial y$. Because of the conditions of Cauchy-Riemann,

$$-d\Phi/dz = -\partial\varphi/\partial x - i\partial\psi/\partial x = -\partial\varphi/\partial x + i\partial\varphi/\partial y = v_x - iv_y$$

which gives equation (4.12). Equation (4.13), if Φ is expressed in terms of φ, reads: $p = p_0 + \rho\partial\varphi/\partial t - \rho v^2/2$, which agrees with equation (1.29). For details on the complex potential, see any textbook on hydrodynamics.

$$\Phi_k(z) = \frac{\Gamma}{2\pi i} \cdot \ln \frac{\pi(z - z_k)}{l} \tag{4.14}$$

where Γ is the circulation of the velocity and l any length. The simple computation of the velocities v_x and v_y from this potential gives the flow about the point $z = z_k$ shown in figure 40. The velocity $v = \sqrt{v_x^2 + v_y^2}$ changes with the distance r from the axis of the cylindrical vortex according to the law

$$v = \Gamma/2\pi r \qquad \rho = \sqrt{x^2 + y^2}$$

similar to the change in the magnetic field about an infinitely long cylindrical wire. If there are several vortices located at different points $z_1(x_1,y_1)$, $z_2(x_2,y_2)$, . . ., $z_k(x_k,y_k)$, the total potential Φ is obtained by summing over-all Φ_k. The potential Φ' of an infinite series of vortices having the circulation Γ' and located at the distance l from each other will now be considered. In figure 41 are shown two such series of vortices. The vortices of the upper series are located at the points $z'_k = x'_k + iy'_k$, where $x'_k = lk$, $(k = 0, \pm 1, \pm 2, \cdot \cdot \cdot)$ and $y'_k = h/2$. Since the potential Φ is determined only with an accuracy up to a constant, it is possible in equation (4.14) under the logarithmic sign to divide by any number so that the sum Φ_k may be written in the form

$$\Phi'(z) = \frac{\Gamma'}{2\pi i}\left\{\ln\frac{(z - z'_0)\pi}{l} + \sum_{k=1}^{\infty}\left[\ln\left(\frac{z - z'_k}{-lk}\right) + \ln\left(\frac{z - z'_k}{lk}\right)\right]\right\} = \frac{\Gamma'}{2\pi i}\ln\left\{\frac{(z - z'_0)\pi}{l} \times \prod_{k=1}^{\infty}\left(1 - \left(\frac{z - z'_0}{l \cdot k}\right)^2\right)\right. \tag{4.15}$$

and, because of the known representation of the function $\sin(\pi z)$

$$\sin \pi z = \pi z \cdot \prod_{k=1}^{\infty}\left(1 - \frac{z}{k^2}\right) \tag{4.16}$$

there is obtained

$$\Phi' = \frac{\Gamma'}{2\pi i} \ln \sin \frac{\pi}{l} (z - z_0') \tag{4.17}$$

In the same way, the following expression is obtained for the second series of vortices:

$$\Phi'' = \frac{\Gamma''}{2\pi i} \ln \sin \frac{\pi}{l} (z - z_0'') \tag{4.18}$$

The velocity of motion of the vortex chains will now be determined. It is readily seen that one chain cannot move. In fact, all the vortices are under the same conditions and for computing the velocity of motion of the chain it is sufficient to compute the velocity of any of the vortices. The latter is equal to the velocity resulting from all vortices except the one considered since one vortex filament does not by itself give a forward velocity. In view of the symmetry, however, it is evident that the vortices situated on the right and left sides of the vortex under consideration impart to it equal and opposite velocities (this is easily verified by the equation $v = -d\Phi'/dz$, if the potential of the vortex under consideration is subtracted from equation (4.17)). In the presence of two chains, conditions are different. In this case the total potential is given by

$$\Phi = \Phi' + \Phi'' = \frac{\Gamma'}{2\pi i} \ln \sin \frac{\pi}{l} (z - z_0') + \frac{\Gamma''}{2\pi i} \ln \sin \frac{\pi}{l} (z - z_0'') \tag{4.19}$$

and the complex velocity will be

$$v_x - iv_y = -\frac{d\Phi}{dz} = -\frac{\Gamma'}{2\pi i} \cot \frac{\pi}{l} (z - z_0') - \frac{\Gamma''}{2\pi i} \cot \frac{\pi}{l} (z - z_0'') \tag{4.20}$$

The vortices of each of the chains will move in the identical fashion; but since the chain itself does not move, the first chain will be displaced only under the effect of the second, while the second only under the effect of the first. The velocity of the first chain is therefore obtained if the velocity produced by the second chain is computed at the point where some vortex of the first is located (for example, $z = z_0'$). The velocity of the first chain will thus be

$$V_x' - iV_y' = -\frac{\Gamma''}{2\pi i} \cot \frac{\pi}{l} (z_0' - z_0'') \tag{4.21}$$

and of the second

$$V_x'' - iV_y'' = +\frac{\Gamma'}{2\pi i} \cot \frac{\pi}{l} (z_0' - z_0'') \tag{4.22}$$

In order that the chains move without change of the relative configuration, it is necessary that

$$V'_x - iV'_y = V''_x - iV''_y \tag{4.23}$$

that is, $\Gamma'' = -\Gamma'$. If it is desired that the direction of the circulation correspond with that shown in figure 41, it is necessary to take $\Gamma'' = \Gamma > 0$. Assuming further that the chains move parallel to themselves, it is required that $V'_y = V''_y = 0$. This condition permits determining the magnitude of the shift of the vortices of one chain relative to the vortices of the other. When this shift is denoted by b and the distance between the chains by h (see fig. 41), the expression $z'_0 - z''_0 = b + ih$ is obtained. In order to determine b, it is necessary to equate V'_y to zero, that is, to the imaginary part of equation (4.21) or equation (4.22). For this it is necessary to make use of the equation

$$\cot (X + iY) = \frac{\sin 2X}{\cosh 2Y - \cos 2X} - i \frac{\sinh 2Y}{\cosh 2Y - \cos 2X} \tag{4.24}$$

where

$$\cosh \xi = (e^{\xi} + e^{-\xi})/2, \quad \sinh \xi = (e^{\xi} - e^{-\xi})/2, \quad \tanh \xi = \sinh \xi/\cosh \xi$$

After simple reductions there is obtained from the condition $V'_y = V''_y = 0$

$$\sin \frac{2\pi b}{l} = 0 \tag{4.25}$$

that is, $b = 0$ or $b = l/2$. In the first case the vortices of the two series are one above the other; in the second case they are arranged in chess order as shown in figure 41. Kármán and Rubach (ref. 48) have shown that the first arrangement ($b = 0$) is not stable, while the second one ($b = l/2$) is stable for a wide class of disturbances if

$$\cosh \left(\frac{\pi h}{l}\right) = \sqrt{2} \tag{4.25'}$$

that is,

$$\frac{h}{l} = 0.28$$

In this way the ratio h/l is determined. The obtained picture of the disposition and motion of the vortices very nearly corresponds with what is observed in tests on the flow about cylinders and plates (see fig. 36). In particular, experiment confirms the value of the ratio h/l given here.

When the real part V'_x of the complex velocity $V'_x - iV''_y$ (eq. (4.21)) is computed for $b = l/2$, the velocity of motion of the vortex street is obtained

$$u = V'_x = \frac{\Gamma}{2l} \tanh \left(\frac{\pi h}{l}\right) = \frac{\Gamma}{2\sqrt{2l}} \tag{4.26}$$

With the aid of the law of the conservation of momentum applied to the approaching flow, the body, and the vortex street behind the body, Kármán and Rubach (ref. 48; see also ref. 49, the hydromechanics book by Kochin, Kibel, and Roze) succeeded in establishing a relation between the coefficient of head resistance of a body C_w and the ratios l/d and u/v, where d is the diameter of the cylinder or the width of the plate and v is the velocity of the body. At the same time, they identify the street arising behind the body with the infinite vortex street just considered (fig. 42). Very good agreement is obtained with test results as illustrated in the following table:

Body	u/v	l/d	h/l		C_w	
			Theory	Experiment	Theory	Experiment
Cylinder	0.14	4.3	0.28	0.28	0.91	0.90
Plate	.20	5.6	.28	.30	1.61	1.56

The determination from this table of the ratio u/v permits also computing the Strouhal coefficient $\varkappa$ in equation (4.5) for the frequency of the vortex sound for the cylinder and the plate. Thus, in a system of coordinates in which the body is at rest, the vortex street moves with a velocity, equal in absolute value to $(v - u)$, in a direction opposite to the motion of the body (fig. 42). When the vortex street is displaced by l, the entire picture of the vortex motion repeats itself. Hence the period of the motion is $T = l/(v - u)$ and the (fundamental) frequency will be $f = (v - u)/l$. After each time interval T, there occurs behind the body a new completely developed vortex pair. Since it is by these vortices that the vortex sound is generated, the frequency of the vortex sound should be equal to

$$f = \frac{v - u}{l} = \left(1 - \frac{u}{v}\right) \frac{d}{l} \frac{v}{d} \tag{4.27}$$

whence

$$\varkappa = \left(1 - \frac{u}{v}\right) \frac{d}{l} \tag{4.28}$$

When the values u/v and l/d are substituted from the table given previously, $\varkappa = 0.20$ (for the cylinder) and $\varkappa = 0.14$ (for the plate) are obtained, which are in very good agreement with the experimental data previously given. In this way the theory of Kármán and Rubach connects the computation of the head resistance of a body with the computation of the formation of vortices arising behind the body.

24. Pseudosound. Conditions of Radiation of Sound by a Flow

In practice it is often necessary to deal with a sound receiver under conditions where the receiver is immersed in an unsteady flow,

that is, in a flow the pressure and velocity of which vary not only in space but in time. Examples of such flows are a wind which constitutes a turbulent flow possessing a certain mean velocity, the stream of water in a ship's wake separating behind a ship or from some projecting part of its body, and so forth. An idealization of such wake is the Kármán vortex street which moves with the velocity $u = \Gamma / 2\sqrt{2}l$ so that the pressure and velocity of the flow at each point vary in time with the period $T = l/u$. The changes in pressure and velocity of the pressure fluctuations in the sound receiver are generally considered as acoustic interferences. From this point of view the subject will later be considered in the section on the wind shielding of receivers. For the present, however, the problem will be presented with special interest placed upon those sounds which are produced by this flow in the receiver. If the frequency of these impulses is sufficiently large, the receiver in such unsteady flow will "hear" a sound (or noise, depending on the spectral composition of these pulsations). Here those additional sounds which may arise from the vortex formation in the sound receiver itself are neglected with the receiver being assumed ideal in this respect. The effect of the pulsations existing in the flow (on the receiver) may be inseparable from the effect of a sound of similar spectral composition. In both cases, the receiver will receive a sonic effect. The sonic vibrations of the medium, however, and the pulsations of the unsteady flow are physically widely different. In the first case it is a question of the small changes of state of the medium associated with its compressibility. The sonic vibrations are propagated with the velocity of sound, and this velocity is determined by the elasticity of the medium ($c^2 = dp/d\rho$). In the case of pulsations in an unsteady flow, the compressibility (if the velocities in the flow are much less than the velocity of sound) plays an entirely secondary role. The motion of the fluid may be assumed as entirely incompressible and still the pulsations of pressure and velocity can take place and will be received by the receiver as a changed pressure. The velocity of propagation of these pulsations bears no relation to the velocity of sound and is equal to the mean velocity of the flow. The second difference lies in the fact that the sound waves are subject to the principle of superposition (because they may be assumed as linear vibrations of the medium), whereas the pulsations of velocity and pressure in an unsteady flow represent a twofold nonlinear phenomenon and are not, of course, subject to the superposition principle. These physical differences make it necessary to term the sound received by a receiver immersed in an unsteady flow a "pseudosound." It should be borne in mind that an unsteady flow may be the cause of the occurrence of the usual sound propagated with velocity c. An example of this may be provided by the same vortex sound which arises in the flow about bodies. The conditions under which sound is produced by a flow will be considered subsequently.

What has herein been said with regard to the sound of an unsteady flow may easily be illustrated by the example of the Kármán vortex street, which may be considered as one of the simplest schemes of nonsteady flow. For this purpose, the velocity and pressure in a Kármán vortex street will be computed. The pressure receiver will be assumed at rest, so that the computation will be carried out in a system of coordinates in which the vortex street moves with the velocity $u = \Gamma/2\sqrt{2}l$. In this system of coordinates, the coordinates z_0' and z_0'' (of eqs. (4.17), (4.18), and (4.19)), distinguishing the positions of the vortices of the first and second street, will be functions of the time

$$\left.\begin{aligned} z_0' &= ut + i\,\frac{h}{2} \\ z_0'' &= ut - i\,\frac{h}{2} \end{aligned}\right\} \tag{4.29}$$

From equations (4.19) and (4.20) there follows:

$$\frac{\partial\Phi}{\partial t} = \frac{\partial\,\Phi}{\partial z_0'}\,u + \frac{\partial\,\Phi}{\partial z_0''}\,u = -\,u\,\frac{d\Phi}{dz} = u\,(v_x - iv_y) \tag{4.30}$$

The pressure p, on the basis of the Bernoulli's equation (4.13), is equal to

$$p = p_0 + \rho u v_x - \rho\,\frac{v^2}{2} \tag{4.31}$$

Further, from equation (4.20) for $\Gamma' = -\Gamma'' = -\Gamma$ we have

$$v_x - iv_y = \frac{\Gamma}{2\pi i}\cot\frac{\pi}{l}\,(z - z_0') - \frac{\Gamma}{2\pi i}\cot\frac{\pi}{l}\,(z - z_0'') \tag{4.32}$$

When equation (4.24) is used for the determination of the real and imaginary parts, the following expressions are obtained from equations (4.32) and (4.29):

$$v_x = \frac{\Gamma}{2l}\cdot\left\{\frac{\sinh\frac{2\pi}{l}\left(y + \frac{h}{2}\right)}{\cosh\frac{2\pi}{l}\left(y + \frac{h}{2}\right) + \cos\frac{2\pi}{l}\,(x - ut)} - \frac{\sinh\frac{2\pi}{l}\left(y - \frac{h}{2}\right)}{\cosh\frac{2\pi}{l}\left(y - \frac{h}{2}\right) - \cos\frac{2\pi}{l}\,(x - ut)}\right\} \tag{4.33}$$

$$v_y = \frac{\Gamma}{2l} \sin \frac{2\pi}{l} (x - ut) \left\{ \frac{1}{\cosh \frac{2\pi}{l} \left(y + \frac{h}{2}\right) + \cos \frac{2\pi}{l} (x - ut)} - \right.$$

$$\left. \frac{1}{\cosh \frac{2\pi}{l} \left(y - \frac{h}{2}\right) - \cos \frac{2\pi}{l} (x - ut)} \right\} \tag{4.33'}$$

In view of the complexity of the equations, the pressure p will be determined for two extreme cases: (a) on the axis of the street ($y = 0$), and (b) outside the street, for $y > l > h$. In the first case from equations (4.31), (4.33), and (4.33'), the following expression is obtained when $\Gamma/2l = \sqrt{2 \cdot u}$ and $\cosh(\pi h/l) = \sqrt{2}$ are taken into account:

$$p = p_0 + \frac{4\rho u^2}{1 - \dfrac{\cos^2 \frac{2\pi}{l} (x - ut)}{\cosh^2 \frac{\pi h}{l}}} \left\{ 1 - \frac{4 + 2 \sin^2 \frac{2\pi}{l} (x - ut)}{1 - \dfrac{\cos^2 \frac{2\pi}{l} (x - ut)}{\cosh^2 \frac{\pi h}{l}}} \right\} \tag{4.34}$$

From this equation it is seen that the amplitude of the variable pressure on the axis of the chain is of order of magnitude equal to ρu^2, and the fundamental frequency of vibration of this pressure is $\omega = 4\pi u/l = 2\omega_0$ ($\omega_0 = 2\pi u/l$ is the fundamental frequency of the chain). For large y, the terms of the order $(e^{-2\pi y/l})^2$ and higher being neglected, there is obtained

$$p = p_0 - 4\rho u^2 e^{-\frac{2\pi y}{l}} \sqrt{2} \cos \frac{2\pi}{l} (x - ut) + \cdot \cdot \cdot \tag{4.35}$$

As is seen from equation (4.35), the pressure gradually approaches p_0, its amplitude is now equal to $4\sqrt{2}\, \rho u^2 e^{-2\pi y/l}$, and the fundamental frequency $\omega = \omega_0$. The spectrum of the vibrations depends essentially on the position of the receiver in the street; at the depth of the street the predominating frequency of the pseudosound is $2\omega_0$ and outside it, ω_0.

It should be observed that this computation of the pressure refers to an ideal receiver which does not introduce a distortion in the flow. A real receiver situated in a flow unavoidably changes it near the body of the receiver. The pressure received by the receiver will depend not only on what occurs in the flow itself but also on the character of the flow about the receiver. For this reason it is necessary to take into account the precise manner in which the pressure distribution of the flow changes when the receiver is introduced.

Since the computation of such distortions is practically nonrealizable, the discussion must be limited to approximations. The equations of hydrodynamics

$$\rho\left\{\frac{\partial\vec{v}}{\partial t} + (\vec{v}, \nabla\,\vec{v})\right\} = -\nabla p \tag{4.36}$$

permit writing down for the pressure the following equation, which is valid for the order of magnitude considered:

$$p = \alpha\rho\cdot\frac{\delta v}{T}\,l + \beta\rho v\delta v + \text{constant} \tag{4.37}$$

where δv is the magnitude of the velocity pulsations in the flow, T the period of these pulsations, α and β numerical coefficients, and l a length determining the gradients (for example, $\nabla p \sim p/l$). If the linear dimension of the pulsations is Λ, $T = \Lambda/v$. In the flow itself, evidently, $l = \Lambda$, so that both sides in equation (4.37) are of the same order (provided the flow is not near the steady condition) Near the body of the receiver, the characteristic length determining the gradients will be either Λ or a dimension of the receiver d, if $d < \Lambda$. In the first case ($d > \Lambda$), there is obtained from equation (4.37)

$$p = (\alpha + \beta)\rho v\delta v \tag{4.38}$$

in the second ($d \ll \Lambda$):

$$p = \left(\alpha\cdot\frac{d}{\Lambda} + \beta\right)\rho v\delta v \cong \beta\rho v\delta v \tag{4.39}$$

The coefficients α and β depend on the character of the flow and on the shape of the receiver body because it is a function of the pressure near the receiver. An essential conclusion, important for the wind shielding of the receiver, is that in the case where the dimensions of the receiver are very much less than the dimensions of the pulsations the receiver will register the pressure changes which are produced not by the local acceleration $\partial v/\partial t$ but by the change in the aerodynamic pressure $\rho v^2/2$; that is, the situation would be that which would be obtained for a slow change of velocity of a constant flow. In this case, therefore, it is permissible to consider the flow about the receiver as a constant flow and, since the pressure distribution on the receiver is known for such flow

$$p = \varkappa\cdot\frac{\rho v^2}{2} \tag{4.40}$$

(where $\varkappa$ depends on the point of the surface of the receiver), the changed pressure may be computed by the equation

$$\delta p = \varkappa\,\rho v\delta v \tag{4.41}$$

(δv is the pulsation of velocity). In the case $d > \Lambda$, such simplified

"quasistationary" consideration is no longer possible. This case will be considered in more detail in sections 29 and 30.

The pressure pulsations computed previously, which have been termed pseudosound, are due to the motion of an incompressible fluid. The compressibility could have been taken into account as a further small correction of the order of u^2/c^2. The question may be raised, however, whether it is possible that behind these small corrections there is nevertheless hidden a true sound propagated with its characteristic velocity c. The answer to this question must be given in the negative. If the receiver moves together with the street, that is, with velocity u, all the magnitudes will become constant. In such a system of coordinates, the flow of the vortex street becomes stationary. It is now shown that if there exists a system of coordinates in which the flow is stationary, such flow cannot radiate sound. The possibility of reducing the flow to a stationary form means that all the magnitudes characterizing the flow depend on the time only through the combination $x - ut$, so that by taking the new system moving with velocity u $(x' = x - ut)$ a stationary flow is obtained. Hence, the potential of the velocities Φ will likewise be a function of $(x - ut)$ (even if the compressibility of the fluid is taken into account); that is,

$$\Phi = \Phi(x - ut,y,z) \tag{4.42}$$

This potential is expanded into a system of cylindrical waves passing off from the flow

$$\Phi(x - ut,y,z) = \int C(\alpha,\beta)e^{i\alpha(x-ut)}H_0(\beta\rho)\beta d\beta d\alpha \tag{4.43}$$

where $\rho = -\sqrt{x^2 + z^2}$ is the distance from the axis of the flow, H_0 is a Hankel function, and α and β are parameters of the expansion. Each of the individual cylindrical waves

$$\Phi_{\alpha\beta}(x,\rho) = C \cdot e^{i\alpha(x-ut)} \cdot H_0(\beta\rho) \tag{4.44}$$

for large ρ assumes the asymptotic form:

$$\Phi_{\alpha\beta}(x,\rho) = C \cdot \frac{e^{i\alpha(x-ut)+i\beta\rho}}{\sqrt{\rho}} = \frac{C}{\sqrt{\rho}} \cdot e^{i(\alpha x+\beta\rho)-i\omega t} \tag{4.45}$$

where the frequency $\omega = \alpha u$. But for sound waves the phase velocity is c; hence for these waves

$$\alpha^2 + \beta^2 = \frac{\omega^2}{c^2} = \alpha^2 \frac{u^2}{c^2} \tag{4.46}$$

or

$$\beta^2 = \alpha^2 \left(\frac{u^2}{c^2} - 1\right) \tag{4.47}$$

It follows that if $u < c$, then β is imaginary and therefore the vibrations $\Phi_{\alpha\beta}$ damp out exponentially with increasing distance from the flow; in other words, for a flow velocity less than the velocity of sound, the sonic field in the wave region $(|\beta|\rho \gg 1)$ is equal to zero.

For supersonic velocity $(u > c)$, equation (4.47) is possible also for real β. Since $\beta/\alpha = \tan \varepsilon$, where ε is the angle between the normal to the wave and the flow velocity, the following expression is obtained for $u > c$ (eq. 4.47)):

$$\sin \varepsilon = \frac{u}{c} \tag{4.48}$$

that is, radiation is possible only under the Mach angle. This result has already been obtained by a different method (cf. section 20). From this it is seen that a flow moving with subsonic velocity may radiate only in the case where it cannot be made stationary in any system of coordinates. As a particular case, it then follows that the infinite Kármán vortex street cannot radiate sound. Its entire field, even in the case of large frequencies u/l, will be pseudosonic.

25. Vortex Sound in the Flow about a Long Cylinder or Plate

The occurrence of vortex sound in the flow about a body of simple shape, such as a cylinder or plate, is now considered in more detail. Figure 42 shows a section of the cylinder under consideration and the vortex street obtained behind it. A system of coordinates ξ,η,ζ will be taken in which the body is at rest while the air moves with velocity v in the direction of the ξ axis. If in the stationary system of coordinates the velocity of the vortices in the street is u, this velocity in the chosen system of coordinates will be $V = v - u$ (cf. section 23). The continued existence of the Kármán street is maintained by the periodical separation of vortex filaments from the edges of the body in the flow. If the period of the street is denoted as previously by l, the frequency of the vortex separations, as has already been explained in section 23 (cf. eq. (4.27)), is equal to

$$f = \frac{v - u}{l} = \varkappa \cdot \frac{v}{d} \tag{4.49}$$

At the distance, for example, $l/2$, from the body the formed vortices turn into a regular Kármán street and move on uniformly with velocity $V = v - u$. Hence the state of the flow in this region $(\xi > l/2)$ will depend on the time through the combination $\xi - Vt$; and in agreement

with what was said in section 24, the occurrence of vortex sound will be entirely due to the periodic generation of vortices about the body (the region to the left of O'O" in fig. 42). It has already been shown (section 22) that the wave length of a vortex sound $\lambda = \frac{1}{\varkappa}\frac{c}{v}d$ is much greater than the dimensions of the body d. Because of this circumstance, a region can be drawn about the body in which the motion of the fluid may be considered as incompressible, the more so as the distance from it is increased, so that the linear equation for the potential φ, considered in section 6 (see eq. (1.94)) may be considered as valid

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} - \frac{2\beta}{\sqrt{1-\beta^2}}\frac{1}{c}\frac{\partial^2\varphi}{\partial t \cdot \partial\xi^*} = 0 \tag{4.50}$$

where

$$\xi^* = \xi/\sqrt{1-\beta^2}, \quad \xi = x - vt, \quad \beta = v/c$$

The linearity of the phenomenon is attained in that region where the second term of the Bernoulli equation $p = \rho\partial\varphi/\partial t - \rho(\nabla\varphi)^2/2$ may be neglected. If, for the purpose of an approximation, the relations existing in the Kármán street are considered, it may be assumed on the basis of equation (4.35) that the second term $\rho(\nabla\varphi)^2/2$ is small in comparison with the first for the distances from the axis of the street H satisfying the inequality $1/2\sqrt{2} \gg e^{-2\pi H/l}$, which is the case for $H \cong l/2$. Actually the vortices do not break away from the body in an entirely regular manner so that in the spectrum of the vortex sound there are present, in addition to the fundamental Strouhal frequency f, also other frequencies (upper harmonics of f and sound noises). Consideration is restricted to the fundamental frequency f which dominates in intensity and corresponds to the formation of the ideal Karman street. For this reason, only that part of the potential φ will be considered which depends harmonically on the time with frequency $f = \varkappa v/d$. The corresponding wave number $2\pi f/c$ will be denoted by $k(k = 2\pi f/c = \omega/c)$. A certain control surface S is now passed about the body and the Kármán street so that near the body it goes through the previously mentioned region where, on the one hand, equation (4.49) holds and, on the other hand, the motion of the fluid may be considered as incompressible. Ahead of the body, this surface will be considered a plane AB continued by the planes AC, BD (see fig. 42) covering the vortex street. Further, it is evidently sufficient to consider a segment of a cylinder of length $(-L/2 < \zeta < L/2)$ since the state along the cylinder does not vary and the end effects will be neglected. If the values of the potential and its derivatives on this surface are known, applying the theorem of Kirchhoff (eq. (1.103)) generalized for equation (4.49) gives the value of the potential at any point of space. The integral over the chosen surface breaks up into two essentially different parts: the integral over the surface O"BAO' lying in the region

without waves, and the integral over the planes O'C and O"D, enclosing the Kármán street and lying in a considerable part of the wave region. The values of the potential φ and its derivatives on the first of the aforementioned surfaces may be replaced by the values Φ_0 representing the motion of an incompressible fluid. The integral over AB then drops out since this surface is drawn through the undistributed flow where $\Phi_0 = 0$, and there remain the integrations over AO' and BO". On the planes O'C and O"D, passing to infinity, the potential φ may be represented in the form of the sum of the potentials Φ_0' and φ'. The first represents the potential of the Kármán street, and its integral vanishes in the wave region (the street does not radiate). The second, φ', represents the part of the wave field due to the shed vortices. The integral of this part will give at a point of observation P a certain, in general nonvanishing, result which will be denoted by φ_P''. If the part of the field at the point P due to the integration over AO' and BO" is denoted by φ_P', the following is obtained for the wave field at P: $\varphi_P = \varphi_P' + \varphi_P''$. Since the surfaces of integration AO' and BO" pass near the source, the integration over them should give the principal part of the field φ_P'. The field φ_P'', however, having the same physical cause as φ_P', cannot possess symmetry other than φ_P' (they are both produced by the same incompressible motion of the fluid near the cylinder). Hence, the magnitude φ_P'' is at least of the same order as φ_P' and has the same symmetry. Therefore, it is sufficient to compute φ_P' for estimating the order of magnitudes and determining the symmetry of the field (zero source or dipole, etc.). This field is obtained by the application of theorem (1.108) to the surface AO' and BO"; that is,

$$\varphi_P' = \frac{e^{i\omega t}}{4\pi}\int_{\xi_1}^{\xi_2} d\xi \int_{-L/2}^{L/2} d\zeta \left\{\frac{\partial \Phi_0}{\partial n}\frac{e^{-ikR}}{R^*} - \Phi_0 \frac{\partial}{\partial \eta}\left(\frac{e^{-ikR}}{R^*}\right)\right\}_{\eta=H} -$$

$$\frac{e^{i\omega t}}{4\pi}\int_{\xi_1}^{\xi_2} d\xi \int_{-L/2}^{L/2} d\zeta \left\{\frac{\partial \Phi_0}{\partial n}\frac{e^{-ikR}}{R^*} - \Phi_0 \frac{\partial}{\partial \eta}\left(\frac{e^{-ikR}}{R^*}\right)\right\}_{\eta=-H} \tag{4.51}$$

where ξ_1 and ξ_2 are the coordinates of point A and point O, respectively, so that $\xi_2 - \xi_1 \cong l$ and $H \cong l/2$. It will now be assumed that the symmetry in that region where the vortices are generated for the part of the flow having the principal frequency is the same as the symmetry of the flow in the developed Kárman street; that is, it is assumed that the vortices are developed in alternation, first on the upper edge of the body then on the lower, with a phase shift π (this more descriptive requirement is somewhat more rigid than the requirement just formulated for the component of the motion having the frequency f, but from

the second the first necessarily follows). This symmetry is characterized by the relations[34]

$$\Phi_0(\xi,\eta,\zeta)_{\eta=H} = -\Phi_0(\xi,\eta,\zeta)_{\eta=-H} \tag{4.52}$$

$$\left(\frac{\partial\Phi_0(\xi,\eta,\zeta)}{\partial\eta}\right)_{\eta=H} = \left(\frac{\partial\Phi_0(\xi,\eta,\zeta)}{\partial\eta}\right)_{\eta=-H} \tag{4.53}$$

and leads to a dipole radiation with a dipole the axis of which is perpendicular to the flow (plus higher multiple radiations the intensities of which will decrease $(kl)^2$ times with increase in the multiplicity of the poles). In fact, if in equation (4.51) e^{-ikR}/R^* and $\partial(e^{-ikR}/R^*)\partial\eta$ are expanded in a Taylor series in powers of $kH \ll 1$ and $r \to \infty$, the following expression is obtained from equations (4.52) and (4.53):

$$\varphi_P' = -\frac{e^{i(\omega t-kR)}}{2\pi R^*}\cdot\frac{\partial R}{\partial\eta}\cdot ikH \times$$

$$\int_{\xi_1}^{\xi_2} d\xi \int_{-L/2}^{L/2} d\zeta \left\{\frac{\partial\Phi_0}{\partial\eta} + \frac{\Phi_0}{H}\right\}_{\eta=H} + \ldots \tag{4.54}$$

The magnitudes $\partial\Phi_0/\partial\eta$ and Φ_0/H are proportional to v, and their mean values differ from v only by a factor. The magnitude H is approximately equal to $l/2$, $\xi_2 - \xi_1 \cong l$. Finally, $\partial R/\partial\eta$ differs from $\cos\theta$ only by quantities of the order of v^2/c^2, where θ is the angle between the η-axis (axis of the dipole) and the direction to the point of observation P. In place of equation (4.54), the following may therefore be written:

$$\varphi_P' = -\frac{ik\alpha\cdot v}{4\pi}\cdot\frac{e^{i(\omega t-kR)}}{r}\cdot Ll^2\cos\theta \tag{4.55}$$

where α is a numerical coefficient $\ll 1$ (representing essentially the mean value of the nondimensional velocity $1/v\cdot\partial\Phi_0/\partial\eta$ on the

[34]The symmetries of the Karman street corresponding to these relations are easily verified if it is borne in mind that the potential of one vortex chain extending over the length $y' = h/2$ is symmetrical with respect to the substitution of $-(y - h/2)$ for $(y - h/2)$, and the other, extending over the length $y'' = -h/2$, is symmetrical with respect to the substitution of $-(y + h/2)$ for $(y + h/2)$, and that the phases of the potentials along the x-axis for the fundamental frequency are displaced by a half period.

η-axis in the plane AO). The energy flow $\vec{N}$ in a system of coordinates in which the medium is at rest and the body moves with velocity v is now computed. According to equation (1.60), this flow is equal to

$$\vec{N} = \frac{i\rho\omega'}{4}\left(\varphi^*\nabla\varphi - \varphi\nabla\varphi^*\right) = \frac{\rho\omega' k}{2}\nabla R\left|\varphi\right|^2 \tag{4.56}$$

where ω' is the frequency which has been changed because of the Doppler effect:

$$\omega' = 2\pi f\left(1 - \frac{1}{c}\frac{dR}{dt}\right) = 2\pi f\left(1 - \frac{v}{c}\cos\theta\right) \tag{4.57}$$

Since the quadrupole radiation was neglected, there is no point in retaining higher powers of v/c in substituting φ from equation (4.49) into equation (4.50). Neglecting these terms yields:

$$N_r = \frac{\alpha^2\cos^2\theta}{32\pi^2 r^2}\frac{\rho\omega^4 v^2}{c^3}L^2 l^4 \tag{4.58}$$

and the total energy radiated per second will be

$$I = \int N_r d\Omega = \frac{\alpha^2}{24\pi}\frac{\rho\omega^4 v^2}{c^3}L^2 l^4 \tag{4.59}$$

Noting that $\omega = 2\pi f = \frac{2\pi v}{l}(1 - u/v)$, where u is the velocity of the vortices, gives

$$N = \frac{\pi^2\alpha^2\cos^2\theta}{2r^2}\frac{\rho v^6}{c^3}L^2\left(1 - \frac{u}{v}\right)^4 \tag{4.58'}$$

$$I = \frac{2\pi^3}{3}\alpha^2\frac{\rho v^6}{c^3}L^2\left(1 - \frac{u}{v}\right)^4 \tag{4.59'}$$

where, according to the table given previously, for the cylinder $(1 - u/v) = 0.86$ and for the plate $= 0.80$. The equation obtained earlier from dimensional considerations (see section 22) is again obtained. The direction of the dipole axis is now fixed; however, the axis extends perpendicular to the direction of the flow. It is seen further that the intensity is proportional to the square of the length of the segment of the cylinder (or plate). According to the observations of W. Holle (ref. 46) for small aspect ratios ($L/d \lesssim 15$), the intensity of the sound is proportional to a power of L, near 2. For $L/d > 30$, according to reference 46, I is proportional not to L^2 but to Ld. The essential point evidently is the fact that for large L/d the coherence of the radiation by the individual parts of the cylinder is disrupted. This consideration is very likely if it is remembered that the long vortex filaments, as they are considered in the Kármán theory, are not very stable and break up into certain segments of length ΔL.[35]

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[35] This assumption sould be verified by experimental check.

The intensity will then be proportional not to L^2 but to

$$\sum \Delta L^2 = \frac{L}{\Delta L} \Delta L^2 = L\Delta L$$

where ΔL does not now depend on L so that $\Delta L = \beta d$, where β is a certain numerical coefficient depending in general on d/L. For medium values of the aspect ratio L/d, $\beta = L/d$; and for large values of L/d, β = constant.

Thus in place of equations (4.52) and (4.53), the following will apply for long bodies;

$$N = \frac{\pi^2\alpha^2 \cdot \beta \cos^2\theta}{2r^2} \frac{\rho v^6}{c^3} Ld \left(1 - \frac{u}{v}\right)^4 \tag{4.58"}$$

$$I = \frac{2\pi^3}{3} \alpha^2\beta, \frac{\rho v^6}{c^3} Ld \left(1 - \frac{u}{v}\right)^4 \tag{4.59"}$$

If the results of the previously mentioned tests of Holle are used, it is to be expected that $\beta = L/d$ for $L/d \sim 10$ and β = constant for $L/d > 20$.

Both from the earlier derived equations and from those now obtained, it follows that the intensity of the vortex sound is proportional to the density of the medium ρ and inversely proportional to the cube of the sound velocity. Hence the intensity of the vortex sound, for otherwise equal conditions, is in water 10 times as great as in air. When the intensity in decibels is expressed by the ratio to the threshold pressure 2×10^{-4} bar, there is obtained

$$N(db) = 80 + 10 \log \frac{N\rho c}{4} \tag{4.60}$$

According to the results of W. Holle (ref. 46), the intensity of the vortex sound N is 80 decibels for a cylinder of length L = 22.5 centimeters and diameter d = 1.2 centimeters, for v = 35 meters per second at the distance r = 1 meter (and $\cos\theta = 1$). From these data and equation (4.58'), the value $\pi^2\alpha^2\beta/2 = 10^{-3}$ is obtained, whence for $\beta \cong 10$ there is obtained $\pi\alpha = 1.4 \cdot 10^{-2}$. This value is in good agreement with the initial assumptions of the theory. In fact, α essentially reduces to the value of the ratio v_y/v at the distance $y = l/2$ from the street. According to equation (4.33'), at this distance $v_y/v \cong ue^{-\pi}/v = 0.2e^{-\pi} = 10^{-2}$.

26. Remarks on the Vortex Noise of Propellers

Tests show that the vortex noise of propellers has a spectrum in which one of the frequencies stands out relatively strongly, so that the

spectrum consists of a sharp peak on a diffused background (fig. 43). This characteristic of the vortex noise becomes understandable if account is taken of the fact that its intensity increases very rapidly with the velocity (as v^6). This noise may, in fact, be considered as generated by the vortices shed from the different parts of the blade. A conception of the spectrum of this sound can be obtained if the individual parts of the blade are assumed to give rise to independent vortex formations and if to each part of the blade is applied the equation for the intensity of the vortex sound derived previously from considerations of dimensionality and considered for the special case of a cylinder or plate. The length of a segment of the blade over which the profile and its angle of attack changes little will be denoted by ΔR. The width of the profile at the same segment will be denoted by $l(R)$. The intensity of the vortex sound generated by this segment will then be

$$\Delta I = \gamma l(R)\Delta R v^6(R) \qquad \gamma = \frac{\pi^2\alpha^2\beta \cos^2\theta}{2r^2} \tag{4.61}$$

where $v = 2\pi RN$ is the peripheral velocity of the segment, R is the distance from the axis of the propeller, and N is the number of rotations of the propeller. The frequency which is predominantly radiated by this segment will be

$$f(R) = \varkappa \frac{v(R)}{d(R)} = 2\pi\varkappa N \cdot \frac{R}{d(R)} \tag{4.62}$$

where $d(R)$ is the width of a plate equivalent to the blade element. The following expression may be set up:

$$d(R) = l \sin \alpha + b \cos \alpha \tag{4.63}$$

where α is the angle of attack of the segment, l the width, and b the thickness (l, b, and α are functions of R).

From equations (4.62) and (4.63), the following terms can be found; $R = R(f)$, and also $d(f)$, $l(f)$, $\Delta R(f) = (dR/df)\Delta f$. Substituting in equation (4.61) yields

$$\Delta I = \gamma l(f) d^6(f) f^6 \Delta f \frac{dR}{df} \tag{4.64}$$

which gives the spectral distribution of the vortex sound radiated by the propeller. It has a sharp maximum about a certain frequency f. This is evident from the fact that f and R are approximately linearly connected ($R \sim f$), and for $f \to \infty$, $R \to R_0$, where R_0 is the radius of the propeller (in fact, $d(R_0) = 0$; then from equation (4.62) there follows $f = \infty$). Hence in equation (4.58), the factor f^6 rapidly increases while the factors dR/df and $d^6(f)$ approach 0 as $f \to \infty$.

Equation (4.64) can, of course, give only a very rough idea of the spectral composition and the intensity of the vortex sound of a propeller, since the assumptions made in its derivation do not pretend to

great accuracy. The angle θ entering the coefficient γ, as is known from section 25, is the angle between the ray at the point of observation and the dipole axis which is perpendicular to the flow. Since the blades move perpendicularly to the propeller axis, this is the angle between a ray at the point of observation and the propeller axis. The maximum intensity of the vortex noise will therefore be radiated ahead of and behind the propeller axis, as, in fact, observed (see ref. 47). It should be recalled that the sound of the propeller rotation (cf. section 18) is, on the contrary, radiated in directions almost perpendicular to the propeller axis. The frequencies of these two sounds, as has already been remarked, are likewise different. The frequency of the rotation sound is $f_0 = Nn$ (n, the number of blades, cf. section 18), while the frequency of the vortex sound is equal to

$$f = \varkappa \cdot \frac{2\pi N\overline{R}}{\overline{d}} \tag{4.65}$$

where $\overline{R}$ and $\overline{d}$ are the values of R and d for the most intense frequency. The ratio of the frequencies of these two sounds will be

$$\frac{f}{f_0} = 2\pi\varkappa \cdot \frac{\overline{R}}{\overline{d}} \cdot \frac{1}{n} \tag{4.66}$$

Since $n = 2$ or 3, $\varkappa \cong 0.2$, and R is generally several times (about 6 to 10) times as large as d, the frequency of the vortex sound exceeds the frequency of the sound of rotation by several times.

27. Excitation of Resonators by a Flow

In the preceding sections, the origin of the sound in the flow of air about bodies was considered. This theory cannot, however, be applied directly to bodies of any shape. It was tacitly assumed that the body has a relatively simple geometrical shape capable of being characterized with sufficient completeness by a single length d, which also determines the frequency of the radiated sound by the equation of Strouhal $f = \varkappa v/d$. For bodies of more complicated shape, the case is otherwise. It is clear, for example, that if on a body of simple shape with characteristic dimension d_1 there is a projection with characteristic dimension d_2, there will be two vortex frequencies for the same velocity of the flow v:

$$\left.\begin{aligned} f_1 &= \varkappa_1 \cdot \frac{v}{d_1} \\ f_2 &= \varkappa_2 \cdot \frac{v}{d_2} \end{aligned}\right\} \tag{4.67}$$

The presence of any projections, sharp angles, discontinuities in the profile, roughnesses, and so forth, may essentially change the sound spectrum. Entirely different characteristic phenomena arise in those cases where the body possesses not convexities but concavities. The latter are acoustic resonators possessing proper vibrations with frequencies υ_s and damping coefficients h_s. The proper frequencies of such a resonator are determined by its dimensions d and the velocity of sound c:

$$\upsilon_s = \frac{c}{d} \psi_s \left(\frac{c}{d} \cdot h_s\right) \tag{4.68}$$

where ψ_s is a certain numerical coefficient. The value of the damping coefficient depends further on the viscosity of the air μ and on its thermal conductivity $\varkappa$ (if the thermal conductivity of the walls of the resonator is much greater than the thermal conductivity of the air, then h_s does not depend on it). It may be said that, in the presence of cavities in the body which are capable of resonance, the frequencies that can be associated with the body depend not only on the ratios v/d but also on the ratios c/d. The simplest examples of such resonators will be, for example, pipes open at one or both ends, Helmholtz resonators (in the form of bottles), and so on. All resonators of such kind may easily be made to emit a sound in an air flow by blowing at their mouths. This phenomenon may be on the most diverse scales, from the whistling in the wind of a small cavity of a receiver microphone (wind static) to the catastrophic excitation of the vibrations of an open wind tunnel that may lead to the destruction of the tunnel and buildings[36]. The same phenomenon in the last war was applied by the enemy in the so-called whistling bombs designed for psychological effect. It finds application to other more suitable purposes in military matters. Also, all musical wind instruments and sirens are essentially based on the phenomenon of the excitation of vibrations by an air stream.

In all these cases there may be distinguished two mutually interacting systems: the vortices arising in the flow about the body on the one hand and the resonator on the other. The vortices do not, of course, represent a rigid system and, strictly speaking, their action on the resonator cannot be considered as the action of an external given force. On the contrary, it is to be expected that the vibrations of the resonator have themselves an effect on the formation of the vortices and on their frequency and intensity so that the entire system must be considered as self-vibrating nonlinear system, the state of which is described by

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[36]An open wind tunnel represents a resonator pipe with open ends and curved like a torus. The flow which excites the vibrations is the flow within the tunnel itself, and vortex formation is obtained at the exit of this stream in the working section. Interesting investigations of the vibrations arising in such system have been conducted by S. P. Strelkov (ref. 50).

the velocity v and the damping coefficient of the resonator h_s. From considerations of dimensionality, the following formula may be written for the amplitude of the pressure fluctuations in the resonator:

$$p = \rho \frac{v^2}{2} \varphi \frac{v\upsilon_s}{d}, \frac{h_s}{\upsilon_s} \tag{4.69}$$

In the region of maximum excitation of the resonator (autoresonance), this amplitude should be inversely proportional to the damping coefficient h_s:

$$p_s = \rho \frac{v'^2}{2} \frac{\upsilon_s}{h_s} \varphi_s \left(\frac{v'\upsilon_s}{d}\right) \tag{4.70}$$

where the stroke on the velocity v indicates that the equation holds only for a certain value $v = v'$. The nonlinear phenomena occurring in the systems under consideration cannot, at the present stage of the theory of vortex formation, be considered in more detail mathematically. The computation of the vortices that arise in the flow about a body even in the absence of a resonator is as yet an unsolved problem. It is all the more reason to expect little success in the computation in the presence of a resonator when, for example, there may occur an interaction of the frequencies of vibration of the vortices with the frequencies of the resonator, phenomena which are characteristic for autovibrating systems. It is therefore of interest to know to what extent it may be useful, for practical purposes, to employ a more primitive point of view in which the nonlinear character of the relations between the vortices and the resonator is ignored and the pressure pulsations produced by the vortex formation are considered as a given external force applied to the resonator. It is evident that such a simplified approach to the phenomenon is possible only in the case where the system of vortices has a considerable degree of independence so that the amplitudes and frequencies of this system are essentially determined by the velocity and geometry of the flow and not by the vibrations of the resonator. If such is the case, the nonlinear phenomena, such as the interaction of the frequencies, could be considered relatively unimportant, and it would not be absolutely required to account for such phenomena in approximations intended for obtaining only the most essential information. It is possible also to assume a priori that the case is otherwise, namely, that in the presence of a resonator the vibrations of the vortices as a whole are determined by the vibrations of the resonator interacting with the flow. The problem proposed could be solved only by an experimental method. Experiments on the excitation of resonators by air streams are reported in reference 51. As a resonator there was taken a four-sided tube closed at one end and placed in an air flow the velocity of which could be brought up to 35 meters per second. At the bottom of the tube was a measuring microphone, with the aid of which the pressures of the vibrations arising in the pipe were transmitted. This pipe was readily excited at definite velocities of the stream, emitting a sound with its natural vibration frequencies $\upsilon_s = c(2s + 1)4l$, $s = 1, 2, 3, \cdots$.

A picture of the flows about such a pipe and within it is shown in figure 44. The circulation arising in the pipe, characteristic in general for all concavities in a stream, is very slow and has no essential value for the phenomena of interest here. On the other hand, of extreme importance is the region about the mouth of the resonator where, as in the case of the flow about solid bodies, an unstable dividing boundary (ABC) is formed between the stream and the stagnant region. It is in this boundary that the vortex formation is obtained, which must, therefore, essentially depend on the geometry of the mouth of the resonator. In order to explain the character of the vortex formation aside from the dependence on the presence of the resonator, the resonator was damped by a damper of cotton and netting placed on the bottom of the resonator. The flow around the mouth was thereby practically unchanged and the resonator was, in effect, eliminated. The damping was chosen such that the frequency characteristic of the measuring microphone located at the bottom of the resonator coincided with the frequency characteristic of the microphone itself. In this way it was possible to determine the spectral composition of the pressure pulsations due to the vortex formation at the mouth of the radiator. It was found that the frequencies of the vortices were in accordance with equation (4.67):

$$f_n = \varkappa \frac{v}{d} n \qquad n = 1, 2, 3, \cdot \cdot \cdot \qquad \varkappa = 0.65 \tag{4.71}$$

where d is the length of the side of the mouth of the resonator. The value of the coefficient $\varkappa$ is given for the angle of attack α of 70° (fig. 44) in the neighborhood of which there was observed the excitation of the resonators. In this way the existence of two overtones of the Strouhal frequency was confirmed, which led in section 22 to the generalized formula (4.6). The amplitude of the pressure of these overtones, as was to be expected, is proportional to the square of the flow velocity:

$$P_n = \beta_n \rho \frac{v^2}{2} \tag{4.72}$$

where $\alpha = 70^\circ$, $\beta_1 = 0.055$, $\beta_2 = 0.020$, and $\beta_3 = 0.010$. Figure 45 shows the frequency of the vortices as a function of the flow velocity v. The same figure shows also the natural frequencies of the resonator υ_s indicated by the horizontal lines. At the points of intersection of these lines, that is, for

$$\upsilon_s = f_n \tag{4.73}$$

indicated on the figure by small circles, which are the points of resonance, the excitation of the resonator was to be expected. This was actually confirmed. On removal of the damper, the resonator was excited at the stream velocities v', determined from equations (4.71) and (4.73):

$$v' = \frac{\upsilon_s d}{\varkappa n} \tag{4.74}$$

The existence of the vortices permits, at least as an approximation, computing the pressures arising in the resonator as a result of the action of the vortices. If $q(\omega)$ denotes the coefficient of amplification of the resonator for the frequency $\omega(\omega = 2\pi f)$, the amplitude of the pressure p on the bottom of the resonator will be as follows, if a pressure of frequency ω and with amplitude P is applied at its mouth:

$$p = q(\omega)P \tag{4.75}$$

This equation assumes that the vibrations are linear; $q(\omega)$ depends on the shape of the resonator, but for all resonators in the region of resonance frequencies $(\omega = 2\pi\upsilon_s)$, $q(\omega)$ is inversely proportional to the damping coefficient h_s. Comparison of the results of computation by this equation with the measured values of p shows (ref. 51) agreement in the order of magnitude. The observed difference attains 6 decibels (2 times), which already serves as an indication of the fact that the divergences are due not to the errors of measurement but to the fact that the assumption of a rigid vortex system fails to correspond; such a system is actually subject to the inverse effect of the vibrations of the resonator (autovibrating character of the phenomena). In figure 46 are given the excitation curves of a resonator (I as a function of v).

The maximums of the excitation correpsond to the resonances of the vortex frequencies and the natural frequencies of the resonator. They are indicated by the same letters as the circles in figure 45. The last maximum c corresponds simultaneously to two resonances (fig. 46), when the second overtone of the vortices coincides with the first overtone of the resonator and simultaneously the fundamental tone of the resonator coincides with the fundamental tone of the vortices[37]. The vibrations that arise in this case are biharmonic.

The height of the maximums is inversely proportional to the damping coefficient, as was confirmed by a change in the damping of the resonator. In the same manner, the dependence of the frequency of the vortices on the dimensions of the mouth of the resonator (eq. (4.71)) is also confirmed. The computation of the amplification coefficient of the resonator for the different resonators is found in many texts on acoustics.

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[37]This circumstance is incidental and is caused by a characteristic feature of the given resonator.

For the simple harmonic Helmholtz oscillator (fig. 47) the equation of vibrations reads:

$$\ddot{\xi} + 2h\dot{\xi} + \omega_0^2\xi = \frac{sP}{M} \tag{4.76}$$

where $\dot{\xi}$ is the velocity of motion of the air mass in the resonator throat, h the coefficient of damping, ω_0 the natural frequency, s the throat area, P the variable pressure applied from without, and M the mass of air moving in the throat of the resonator. Also, $M = \rho Ls$, where L is the effective length of the throat, ρ the density of the air; $L = l + \alpha a$, where l is the length of the throat, a its radius, α a numerical coefficient equal, for the case of a circular opening, to $\pi/2$. The natural frequency ω_0 is equal to:

$$\omega_0 = c\sqrt{\frac{s}{LV}} \tag{4.77}$$

where V is the volume of the resonator. When equation (4.77) is solved for the external force, having frequency ω, the amplitude of the displacement ξ_0 is obtained:

$$\xi_0 = \frac{SP}{M}\frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4h^2\omega_0^2}} \tag{4.78}$$

(if $\omega_0 >> h$). The changes in pressure within the resonator with adiabatic change of the volume of air enclosed in it will have the amplitude:

$$p = \rho c^2 \frac{\Delta V}{V} = \rho c^2 \frac{\xi_0 s}{V} \tag{4.79}$$

Substituting this in equation (4.72) gives

$$p = P\frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4h^2\omega_0^2}} \tag{4.80}$$

whence

$$q(\omega) = \frac{\omega_0^2}{\sqrt{(\omega^2 - \omega_0^2) + 4h^2\omega_0^2}} \tag{4.81}$$

At resonance $(\omega = \omega_0)$, $q(\omega) = \omega_0/2h$, so that the amplitude of vibration of the resonator excited by the flow of air will be

$$p = \beta\rho\frac{v^2}{2}\frac{\omega_0}{2h} \tag{4.82}$$

The numerical coefficient β depends on the shape of the throat and on the angle of approach of the flow (as was mentioned for the rectangular throat, for $\alpha = 70^\circ$, $\beta = 0.055$).

A second simple case of a resonator is a pipe closed at one end, such as that considered in the previously described tests (fig. 44).

In this case it is a question of the vibrations of a distributed system. The displacement of the air along the axis of the pipe, which will be chosen in the direction of the x-axis, is here subject to the wave equation:

$$\frac{d^2\xi}{\partial t^2} + c^2\delta \frac{\partial \xi}{\partial t} - c^2 \frac{\partial^2 \xi}{\partial x^2} = 0 \tag{4.83}$$

where δ is the friction coefficient that takes into account the losses in the heat conduction and viscosity of the air[38].

The pressure p at each point is equal to:

$$p = - \rho c^2 \frac{\partial \xi}{\partial x} \tag{4.84}$$

Equation (17) must be solved for the boundary conditions:

$$(\xi)_{x=0} = 0$$

$$-\rho c^2 \left(\frac{\partial \xi}{\partial x}\right)_{x=l} = P \tag{4.85}$$

expressing that fact that at the closed end of the pipe $(x = 0)$ the air is at rest while at the open end the pressure is equal to the external applied pressure P. If the fact that the air near the mouth of the pipe takes part in the vibrations is taken into account, the last boundary condition must be satisfied by the pressure P', representing the reaction of the associated air mass. If the impedance of this mass, generally termed mouth impedance, is $Z = X + iY$, then $P' = \rho c(X + iY)$. In place of equation (4.85), the following expressions hold:

$$(\xi)_{x=0} = 0$$

$$-\rho c^2 \left(\frac{\partial \xi}{\partial x}\right)_{x=l} = \rho c(X + iY)(\dot{\xi})_{x=l} + P \tag{4.86}$$

The active part of this impedance X is due to the losses in radiation, while the reactive part Y is determined by the mass of the air vibrating along with the resonator. These magnitudes, for an orifice of area s, are equal to:

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[38] A simple method of computing this coefficient is given in reference 4.

$$\left.\begin{aligned} X &= \frac{\omega^2 s}{4\pi c^2} \\ Y &= 0.7 \frac{\omega}{c} \sqrt{\frac{s}{\pi}} \end{aligned}\right\} \tag{4.87}$$

and details on them may be found, for example, in the work of Y. L. Gutin (ref. 52). Assuming that the external pressure depends harmonically on the time, with frequency ω, the displacement ξ is taken proportional to $e^{i\omega t}$. From equation (4.83), a solution satisfying the boundary condition $\xi = 0$ for $x = 0$ is readily found:

$$\xi = \xi_0 \sin Kx \qquad K = \sqrt{\frac{\omega^2}{c^2} - i\omega\delta} \tag{4.88}$$

For small damping ($\delta \ll \omega$) the following may be assumed:

$$K = k - i\varkappa \qquad k = \frac{\omega}{c} \qquad \varkappa = \frac{\delta c}{2} \tag{4.89}$$

By the substitution of equation (4.88) in the second boundary condition, equation (4.86), the amplitude ξ_0 is determined:

$$\xi_0 = - \frac{P}{i\omega\rho c(X + iY) \sin Kl + \rho c^2 K \cos Kl} \tag{4.90}$$

When the real and imaginary parts are separated and the fact that $\varkappa l \ll 1$, $X, Y \ll 1$ is taken into account, the following is obtained, for the amplitude ξ_0:

$$\xi_0 = \frac{P}{\omega\rho c \sqrt{[\cos kl - Y \sin kl]^2 + \left[\sin kl \frac{hl}{c}\right]^2}} \tag{4.91}$$

where

$$h = \frac{c}{l} X + \frac{\delta c^2}{2} \tag{4.92}$$

is the damping coefficient of the radiator.

On the basis of equation (4.84) the amplitude of the pressure at the bottom of the resonator at $x = 0$ is equal to $\xi_0 \rho\omega$. The required amplification factor of the resonator is therefore equal to:

$$q(\omega) = \frac{1}{\sqrt{[\cos kl - Y \sin kl]^2 + \left[\sin kl \frac{hl}{c}\right]^2}} \tag{4.93}$$

The points $\cos kl - Y \sin kl = 0$ determine the position of the resonance frequencies. With the fact that Y is small taken into account, this condition may be represented in the form:

$$\left.\begin{aligned} \cos \frac{\omega L}{c} &= 0 \\ \omega_s &= \frac{\pi c}{2L}(2s + 1), \quad s = 0, 1, 2, \ldots \end{aligned}\right\} \tag{4.94}$$

where L is the effective length of the resonator:

$$L = l + 0.7\sqrt{\frac{S}{\pi}} \tag{4.95}$$

At the points of resonance, the value of the amplification factor is equal to

$$q(\omega) = \frac{l}{hc} \tag{4.96}$$

(since $\sin(k_s l) \cong 1$). Hence the amplitude of the vibrations of the pressure in the resonator under the action of an external stream is:

$$p = \beta\rho \frac{v^2}{2} \frac{l}{h_s c} \tag{4.97}$$

This equation may also serve for estimating the value of p. Consideration will now be given to the computation of the intensity of the sound radiated by a resonator excited by an air stream. Evidently, it is sufficient to compute the energy radiated through the mouth of the resonator and, in what follows, to make use of the law of the inverse square of the distance. The mean flow of energy through the mouth of the resonator according to the general equation (3.3), is equal to:

$$\overline{N} = \overline{p\dot{\xi}}S = \frac{1}{2} p_0'\dot{\xi}_0 \cdot S \tag{4.98}$$

where p_0' is the amplitude of the velocity of the air vibrations near the mouth and $\dot{\xi}_0$ is the amplitude of that part of the air pressure near the mouth which is produced by the radiation. This part is equal to $\rho c X\dot{\xi}_0$. Hence, the mean flow of energy through the entire mouth is

$$N = \frac{1}{2} \rho c \cdot X\dot{\xi}_0^2 S \tag{4.99}$$

and the energy flow through 1 square centimeter at the distance r from the resonator will be

$$N_0 = \frac{1}{8\pi r^2} \rho c X S \dot{\xi}_0^2 \tag{4.100}$$

In order to obtain the final result, the value of $\dot{\xi}_0$ at the point of resonance must be taken. According to equation (4.91), for the tube

$\dot{\xi}_0 = \frac{P}{\rho c} \frac{c}{h_s l}$ and for the Helmholtz resonator, according to equation (4.79), $\dot{\xi}_0 = \frac{P}{\rho c} \frac{c}{2hL}$. Hence, for the tube the following expression applies:

$$N_0' = \frac{X'S'}{8\pi r^2 \rho c} \frac{c^2}{h_s^2 l^2} \beta'^2 \left(\rho \frac{v_s^2}{2}\right)^2 \tag{4.101}$$

and for the Helmholtz resonator:

$$N_0'' = \frac{X''S''}{8\pi r^2 \rho c} \frac{c^2}{2h^2 L^2} \beta'^2 \left(\rho \frac{v_0^2}{2}\right)^2 \tag{4.101'}$$

where v_s is the velocity corresponding to the excitation of the s^{th} vibration of the tube and v_0 is the velocity at which the Helmholtz resonator is excited.

CHAPTER V

ACTION OF A SOUND RECEIVER IN A STREAM

28. Physical Phenomena in the Flow about a Sound Receiver

A sound receiver placed in a stream of air or water will register periodic changes of pressure brought about not only by the arriving sound signal but also by the flow around the body.

Such periodic pulsations are termed "pseudosound." It is clear that the pseudosound will act as an obstacle for the successful reception of the useful signal, an obstacle which may possibly be very significant. It is well known in practice how strongly the audibility of the sound of a distant airplane may be lowered in a wind. Such lowering of the audibility occurs also in the work of hydrophones of a ship (in this case the noises of the ship are intermixed).

For this reason the case of the action of a sound receiver in a stream is of a practical interest. The phenomena due to the unsteady flow must be distinguished from the phenomena that take place in a steady flow. The phenomena that take place in a steady flow are considered first. A steady flow does not contain pressure pulsations periodic in time, but such pulsations arise on the receiver body because of the vortex formation.

The vortex formation is, in the case considered, the only cause of the pseudosound. The predominant frequency of this sound is determined by the formula of Strouhal which was used previously:

$$f = \varkappa \frac{v}{d} \tag{5.1}$$

If the Reynolds number is large $\left(Re = \frac{vd}{\upsilon} > 10^5\right)$, the spectrum of the vortex pseudosound may be very diffuse near the frequency equation (5.1). The pressure of the pseudosound will be proportional to the dynamic pressure:

$$p = \beta \rho \frac{v^2}{2} \tag{5.2}$$

where β is a numerical coefficient that depends on the shape of the body.

If the flow is unsteady, further pressure pulsations characteristic of the flow are superposed on the pressure pulsations determined by the vortex formation. This pseudosound of the flow was partly considered previously (section 24). In this case it is necessary to distinguish between the pressure pulsations brought about by the local change in the velocity of the flow and the pressure pulsations associated with the momentum transfer of the flow. This question was previously discussed in part (section 24), but now it will be considered in greater detail. A simple example may serve to illustrate the pseudosound of the flow. The receiver is assumed to have the shape of a sphere and to be placed in a stream in the direction of the OZ-axis (fig. 48). The flow velocity V is assumed to pulsate periodically with the frequency $\omega = 2\pi/T$: then

$$V = V_0 + \delta V \cdot \cos \omega t \tag{5.3}$$

The vortex formation is disregarded, and the flow is assumed to be potential. The equation for the potential Φ is:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \tag{5.4}$$

The radial component of the velocity $v_r = - \partial\Phi/\partial r$ on the surface of the sphere $(r = a)$ must be equal to zero, and the velocity at a large distance from the body must become $V = - \partial\Phi/\partial z$ (eq. (5.3)). A solution of equation (5.4) satisfying these boundary conditions, as easily verified by substitution, will be

$$\Phi = V \cos \theta \left(r + \frac{a^3}{2r^2}\right) \qquad r \cos \theta = z \tag{5.5}$$

By the formula of Bernoulli, the pressure at such a point will be

$$\frac{p}{\rho} = \frac{\partial\Phi}{\partial t} - \frac{1}{2} (\nabla\Phi)^2 \tag{5.6}$$

and on the surface of the body $(r = a)$, on the basis of equation (5.5),

$$p = \text{constant} + \frac{3}{2} \rho \cos \theta \cdot a \frac{\partial V}{\partial t} - \frac{9}{4} \sin^2\theta \cdot \rho \frac{V^2}{2} \tag{5.6}$$

From this formula, it is seen that the pressure is made up of two component parts, namely, the term p' which is also present in the steady flow:

$$p' = \text{constant} - \frac{9}{4} \sin^2\theta \cdot \rho \frac{V^2}{2} \tag{5.7}$$

and the term p", determined by the acceleration of the flow:

$$p'' = \frac{3}{2} \rho \cos \theta \; . \; a \; . \; \frac{\partial V}{\partial t} \tag{5.8}$$

From a comparison of equations (5.7) and (5.8), it follows from equation (5.3) that the variable part of the pressure p' determined by the dynamic pressure considerably exceeds the part p" determined by the local acceleration if

$$\left.\begin{aligned} a \frac{\partial V}{\partial t} &<< V \, \delta V \\ \text{or} \qquad \frac{2\pi a}{T} &< V \end{aligned}\right\} \tag{5.9}$$

that is, p' << p" if the dimensions of the receiver a are sufficiently small. The velocity pulsation considered is uniform over the entire space. If the pulsations have the dimension Λ, then $T \cong \Lambda/V$, and equation (5.9) reduces to

$$a << \Lambda \tag{5.10}$$

This condition was obtained in section 24 by a different method; it is apparent that, for small dimensions of the receiver in comparison with the dimensions of the pulsations, the changes in dynamic pressure have a much greater significance than the acceleration of the flow.

The spectral distribution of the pseudosound of an unsteady flow is entirely determined by the nature of the flow. If the nonsteady condition of the flow is produced by the flow about certain bodies placed near the receiver so that the receiver is in the vortex street of these bodies, the spectral composition of the pulsations is determined by the Strouhal frequencies and their overtones, as has already been shown in the example of the ideal Kármán street (section 24).

At a large distance from the bodies, the Kármán street undergoes a breakdown and the flow will be turbulent. A natural wind likewise represents a turbulent flow. The fundamental features of this turbulence were described in section 10.

As previously stated in section 24, the computation of the magnitudes and spectral distribution of the pressure pulsations on the surface of a receiver in any unsteady flow is at the present time an insurmountable problem. It was pointed out that a partial analysis of this problem is possible on the basis of dimensional considerations in the application to the fundamental equation of hydrodynamics. In the general case, the pressure on the surface of the receiver is assumed to be determined by

$$p = \alpha\rho a \frac{\partial v}{\partial t} + \beta \frac{\rho v^2}{2} \tag{5.11}$$

This equation is a generalization of equation (5.6), which holds for a particular case. The fact that the pressure at any point and at any instant of time depends on the dynamic pressure $(\rho v^2/2)$ and on the local change in velocity $\partial v/\partial t$ is expressed in equation (5.11). Since the velocity varies not only in magnitude but also in direction, the angle of attack will vary with the velocity fluctuations of the flow. Because of this variance, the numerical coefficients α and β, which depend only on the shape of the body and the angle of attack, will also be functions of the time. If the magnitude of the pulsations δv is much less than the mean velocity of the flow v, the changes in α and β will be slight. Further, the derivative $\partial v/\partial t$ is of order of magnitude equal to $\delta v/T = V\delta v/\Lambda$; and therefore, with equation (5.10) satisfied, $\partial v/\partial t$ may be rejected. For the variable part of the pressure p,

$$p' = \beta_0 \rho v\ \delta v + \left(\frac{\partial\beta}{\partial\psi}\right)_0 \delta\psi\rho\ \frac{v^2}{2} \tag{5.12}$$

where ψ is the angle of attack and the subscript 0 denotes the value of β and $\partial\beta/\partial\psi$ for the angle of attack of the main flow $(\psi = \psi_0)$.

(The variation of the angle of attack $\delta\psi$ is equal to $\delta v_t/v$, where δv_t is the fluctuation of the velocity in the direction perpendicular to v).

For isotropic pulsations, $\delta v_t = \delta v$ and therefore

$$p' = \left(\beta_0 + \frac{1}{2}\left(\frac{\partial\beta}{\partial\psi}\right)_0\right) \rho v\delta v = \boldsymbol{\varepsilon}\, \rho v\delta v \tag{5.13}$$

The spectrum of the pressure p' therefore coincides with the spectrum of the velocity pulsations δv, and its magnitude may be computed from the stationary flow about the body under consideration. This evidently is the only rational conclusion which can be drawn from equation (5.11). For the mean square of the pressure pulsations from equation (5.13)

$$\overline{p'^2} = \varepsilon^2\rho^2 v^2\overline{\delta v^2} \tag{5.14}$$

and the spectral distribution is obtained from

$$\overline{\delta v^2} = \int [\delta v(\omega)]^2 d\omega \tag{5.15}$$

where $\delta v(\omega)$ is the amplitude of the pulsation belonging to the frequency ω. Hence, for the mean-square pressure of all pulsations, the frequencies of which lie between ω_1 and ω_2,

$$\overline{p'^2}(\omega_1,\omega_2) = \varepsilon^2\rho^2v^2\int_{\omega_1}^{\omega_2}\left[\delta v(\omega)\right]^2 q\omega \tag{5.16}$$

The amplification factor of the receiver is assumed equal to $q(\omega)$ in order that the signal received by the receiver is measured by the magnitude P:

$$P = \int q(\omega)p'(\omega)e^{i\omega t}d\omega \tag{5.17}$$

Squaring and averaging equation (5.17) with respect to time yield

$$\overline{P^2} = \int q^2(\omega) \cdot p'^2(\omega)d\omega \tag{5.18}$$

where, on the basis of equation (5.14),

$$p'^2(\omega)d\omega = \varepsilon^2\rho^2v^2\left[\delta v(\omega)\right]^2 d\omega \tag{5.19}$$

If the receiver has sharp resonances to that, for example, there is a natural vibration with the frequency $\omega = \omega_0$ and damping coefficient of h, then for $\omega = \omega_0$ the amplification factor becomes particularly large and, as is known from section 24, is equal to $q(\omega_0) = q'\omega_0/h$, where the coefficient q' is of order of magnitude equal to 1 (section 27). Integrating equation (5.18) with respect to ω_0 between the limits of the resonance line $(\omega_0 - h/2, \omega_0 + h/2)$, yields

$$\overline{p^2(\omega_0)} = \int_{\omega_0-h/2}^{\omega_0+h/2} q^2(\omega_0)p'^2(\omega_0)d\omega = q'^2\frac{\omega_0^2}{h} \cdot p'^2(\omega_0) \tag{5.20}$$

For small values of h, this part of the magnitude $\overline{p^2}$ may predominate over the remaining parts to such an extent that practically the entire effect of the pseudosound on the receiver may be reduced to the emitting of a sound from the receiver at the resonance frequency ω_0. Hence, receivers with sharp resonances will be particularly subject to acoustic disturbances.

This case is characterized by the possibility of reducing the action of the nonsteady flow to that of a steady flow. The fundamental result is the fact that the spectrum of the pressure pulsations reduces to the spectrum of the velocity pulsations. If the approaching flow is a well developed turbulent flow, the approaching flow may be applied to the turbulence theory described in section 10.

This theory was developed for homogeneous isotropic turbulence. The 2/3 law, which determines the spectral distribution of the velocity of the turbulent motion over the pulsations of different scales, was obtained. Now, however, the distribution over the frequencies is of interest. The problem of associating the distribution of the frequencies with the distribution over the spatial scales has been solved only for linear vibrations of the medium (for example, for sonic noise).

In the case of the turbulent motion of a gas or liquid, this relation has not as yet been established.

For the determination of the velocity spectrum over the frequencies, the same considerations which were applied to the diffusion of sound in a turbulent flow (section 12) may be employed. As was explained, the important fact is that the frequency of the turbulent pulsations is in itself very small. The high frequencies, which are of significance in acoustics, are obtained in virtue of the fact that the large-scale velocity pulsations transfer the small-scale pulsations. If the large-scale velocity pulsations which change slowly are included in the mean velocity v (in this way v will have the sense of a mean velocity over a time during which this velocity does not undergo considerable changes and which is much greater than the period of those frequencies which are received by the receiver), then v will be precisely the velocity with which the small velocity pulsations are displaced. For these small pulsations the 2/3 law holds, in accordance with which the mean value of the square of the velocity u for the pulsations which have a scale less than $\Lambda = 2\pi/q$ will be (see section 10, eqs. (2.63) and (2.64)):

$$E(q) = \frac{1}{2} u^2 (q) = \frac{1}{3} \gamma q^{-2/3} \qquad \gamma = \frac{2}{3} \sqrt[3]{2} \left(\frac{D_0}{\varkappa}\right)^{2/3} \tag{5.21}$$

The magnitude $u(q)$ is precisely $\delta v(q)$. Thus,

$$[\delta v(q)]^2 = \frac{2}{3} \gamma \cdot q^{-2/3} \tag{5.22}$$

Since these spatial pulsations of velocity are transferred with velocity v, the frequencies of the corresponding pulsations will be $f = v/\Lambda$ or $\omega = qv$. Thus the intensity of the velocity pulsations with frequencies between ω and ∞ will be

$$[\delta v(\omega,\infty)]^2 = \frac{2}{3} \left(\frac{v}{\omega}\right)^{2/3} = \int_{\omega}^{\infty} [\delta v(\omega)]^2 d\omega \tag{5.23}$$

Differentiating this magnitude with respect to ω yields the required value:

$$[\delta v(\omega)]^2 = \frac{4}{9} \gamma \left(\frac{v}{\omega}\right)^{2/3} \frac{1}{\omega} \qquad (5.24)$$

Substituting this result in equation (5.19) gives the equation for the spectral distribution of the pressure acting on a receiver placed in a turbulent stream.

$$p'^2(\omega)d\omega = \frac{4}{9} \gamma \varepsilon^2 \rho^2 v^2 \left(\frac{v}{\omega}\right)^{2/3} \frac{d\omega}{\omega} \qquad (5.25)$$

The constant γ, as has already been pointed out, is evidently a function of the velocity v. Its value has been considered in connection with the diffusion of sound in a turbulent flow (section 10, eqs. (2.58) and (2.59)).

Since γ and ε may be assumed as known, equation (5.25) permits computing the spectrum of the turbulent noise. As is seen from this equation, the noise of a turbulent flow is concentrated around low frequencies; the intensity of sound near the frequency ω is proportional to $\omega^{-5/3}$. For $\omega = 0$, equation (5.25) is not valid since slow pulsations in the mean velocity v have been included. With regard to the dependence on the wind velocity, if γ is assumed constant, the dependence on the velocity is obtained as $v^{8/3}$. It was pointed out, however, that γ increases with the velocity in a manner which is as yet difficult to determine precisely but which, from all the data available, may be taken approximately as v. If this dependence is taken into account, the noise should increase as $v^{11/3}$. Finally, it must be borne in mind that equation (5.25) is not suitable for high frequencies since in its derivation it was assumed that the dimensions of the receiver were $a \ll \Lambda$. Hence, it is applicable only to $\omega < 2\pi v/a$. (Otherwise, terms are added which are due to the local acceleration $\partial v/\partial t$.) Undoubtedly it gives a lower limit of the sound. The fact that the intensity of the sound increases as $\omega^{-5/3}$ is evidently one of the greatest obstacles to acoustic direction finding, since predominantly low frequencies (80 to 100 hertz) which will be masked by the turbulent sound arrive from a distant airplane.

29. Shielding a Sound Receiver from Vortical Sound Production

No universal method of shielding a sound receiver from vortical sound production is possible. The question depends essentially on the dimensions of the receiver and on the working frequency range, the choice of which is determined by the character of the signal that is to be received. It is nevertheless possible to indicate certain methods that may be found useful.

In the first place, it is possible to vary the dimensions of the receiver so that the Strouhal frequency may be displaced either toward the lower frequencies (by increasing the dimensions of the receiver) or toward the higher frequencies (by reducing the dimensions), depending on the purpose. This method is based on the fact that for the same velocity of the approaching stream and shape of receiver the characteristic Strouhal frequencies are inversely proportional to the linear dimensions of the receiver:

$$\frac{f'}{f''} = \frac{d''}{d'} \tag{5.26}$$

In those cases where the change in dimensions of the receiver is not a rational procedure, a sound-transparent screen F of netting or fabric (fig. 49) may be applied. The principal air flow in this case tends to pass around the screen and the velocity of the flow within is considerably lowered. The Strouhal frequency formed on the vortex deflector is then lowered and will be equal to

$$f' \cong f \cdot \frac{d}{D} \tag{5.27}$$

where f is the Strouhal frequency on the body of the receiver M, d the dimension of the receiver, and D the dimension of the deflector. It is necessary to avoid angles, projections, and so forth, on the body of the deflector, since they may become the cause of vortex formation in an undesirable range of frequencies.

In addition to the effect of lowering the frequency, which was due to the screen, the region of the vortex formation is farther removed from the body of the receiver, another useful result of this method. A part of the flow will nevertheless pass through the screen, but its velocity v' will be less than the velocity of the approaching flow v. Because of this lower velocity, the frequency of the vortex formation immediately at the receiver is likewise lowered in the ratio

$$f'' = f \cdot \frac{v'}{v} \tag{5.28}$$

while the amplitude of the pressure will drop by $(v'/v)^2$. The value of v can not be computed exhaustively, but reasonable estimates may be made. For this purpose, the resistance of the nettings to the flow of air (or of water) must be considered.

In view of the fact that nettings or screens are widely applied for various wind-shielding apparatuses, it is necessary to discuss them in more detail. If the difference in pressure on both sides of the screen is set equal to Δp, and the volume rate of flow of the air through it is set equal to Q cubic centimeters per second, then

$$\Delta p = WQ \tag{5.29}$$

The magnitude W defined by this equation is the resistance of the screen. This magnitude may be represented in the form

$$W = w \frac{l}{S} \tag{5.30}$$

where S is the cross-sectional area of the flow, l the effective thickness of the screen (or other porous partition, for example, fabric), and w the resistivity. The characteristic magnitude for a screen is, of course, the product wl. The resistance W is considered as the resistance of a system of parallel ducts (or small tubes), the length of which is equal to l and the cross-sectional area of which is equal to σ; the area of the screen on which, on the average, there is one opening is denoted by Σ. If the ducts are not identical, then l, σ, and Σ must be considered as the characteristics of the mean representative of the ducts. In order to determine the pressure drop Δp over the length of the duct l, the equations of Navier-Stokes reduced to nondimensional form were used. For measuring the coordinates along the duct, the length scale was taken as the length of the duct l, and for the transverse scale, the magnitude $\sqrt{\sigma}$. The scale of velocity would be the velocity of the flow u, and the scale of the acceleration the magnitude ωu, where ω is the frequency of the flow pulsations. These equations are:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v},\nabla)\vec{v} = -\frac{\nabla p}{\rho} + \upsilon \cdot \nabla \vec{v} \tag{5.31}$$

where $\upsilon = \mu/\rho$ is the kinematic viscosity. The derivatives $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ may be reduced to the derivatives along the duct $(\partial/\partial s)$ and transverse to the duct $(\partial/\partial n)$. Then, setting $\partial/\partial t = \omega\partial/\partial t'$, $\partial/\partial s = (1/l)\partial/\partial s'$, $\partial/\partial n = (1/\sqrt{\sigma})\partial/\partial n'$, $\vec{v} = u\cdot\vec{v}'$, and $p = \Delta p\cdot p'$, equation (5.31) reduces to the form:

$$\omega u \frac{\partial \vec{v}'}{\partial t'} + u^2\left(\frac{v'_s}{l}\frac{\partial \vec{v}'_s}{\partial s'} + \frac{v'_n}{\sqrt{\sigma}}\frac{\partial \vec{v}'}{\partial n'}\right)$$

$$= -\frac{\Delta p}{l\rho}\left(\vec{s}\,\frac{\partial p'}{\partial s'} + \frac{l}{\sqrt{\sigma}}\,\vec{n}'\,\frac{\partial p'}{\partial n'}\right) + \frac{\nu u}{\sigma}\left(\frac{\partial^2 \vec{v}'}{\partial n'^2} + \frac{\sigma}{l^2}\frac{\partial^2 \vec{v}'}{\partial s'^2}\right) \tag{5.31'}$$

where all the stroked magnitudes are nondimensional and the magnitudes and their derivatives are of the same order; $\vec{s}'$ and $\vec{n}'$ are unit vectors along and transverse to the duct. The case where the viscosity is the predominant factor is considered first. In this case the last term predominates over the others. Dividing the entire equation by $\upsilon u/\sigma$, the desired pressure drop Δp will be measured in the units $l\rho\upsilon u/\sigma$ ($8\pi\mu l u/\sigma$ could be used as this corresponds to the Poiseuille law).

in equation (5.31') dimensionless parameters on which Δp may depend, namely, $\omega\sigma/\upsilon$, $u\sigma/\upsilon l$, and σ/l^2, will enter. Thus,

$$\Delta p = \frac{8\pi\mu l u}{\sigma} F_1 \left(\frac{u\sigma}{\upsilon l}, \frac{\sigma}{l^2}, \frac{\omega\sigma}{\upsilon}\right) \tag{5.32}$$

where F_1 is a certain dimensionless coefficient depending on the indicated parameters. The term $u\sigma/\upsilon l$ represents the Reynolds number and determines the ratio of the inertia to the viscosity forces. The value of this term is small for small u, but for these values of u the equation becomes linear; hence for small velocities of flow F_1 practically does not depend on $u\sigma/\upsilon l$. Further, for long ducts $(\sigma/l^2 \ll 1)$ the pressure drop should be proportional to the length of the duct l; hence, F_1 should likewise not depend on σ/l^2. Thus,

$$\Delta p = \frac{8\pi\mu l}{\sigma} u\psi \left(\frac{\omega\sigma}{\upsilon}\right)$$

for

$$\frac{u\sigma}{l} \ll 1, \quad \frac{\sigma}{l^2} < 1 \tag{5.32'}$$

Finally, for small frequencies $(\omega\sigma/\upsilon \ll 1)$ the Poiseuille law must be obtained so that $\psi(0) = 1$. For large $\omega\sigma/\upsilon$, the coefficient $\psi \cong \sqrt{\omega\sigma/\upsilon}$ (see, e.g., Crandall, ref. 53). For $u\sigma/\upsilon l, > 1$, the forces of inertia will predominate over the viscous forces, and therefore it is convenient to use the dynamic pressure $\rho u^2/2$ as the measure of the pressure. In an analogous manner, the following equation is obtained in place of equation (5.32):

$$\Delta p = \frac{\rho u^2}{2} \cdot \frac{l}{\sqrt{\sigma}} \cdot F_2 \left(\frac{\upsilon l}{u\sigma}, \frac{\sigma}{l^2}, \frac{\omega\sqrt{\sigma}}{u}\right) \tag{5.33}$$

For small values of the parameters entering the function F_2 this coefficient will only slightly depend on them. The square law of the resistance is therefore obtained. The effect of the acceleration is now determined by the parameter $\omega\sqrt{\sigma}/u$. The volume rate of flow Q is computed as

$$Q = u\sigma \frac{S}{\Sigma} \tag{5.34}$$

When u from equation (5.32) is substituted and when equation (5.34) is compared with equation (5.29),

$$W_0 = \frac{8\pi\mu}{\sigma} \frac{\Sigma}{\sigma} \psi \left(\frac{\omega\sigma}{\upsilon}\right) \frac{l}{S}$$

for

$$\frac{u\sigma}{\upsilon l} \gg 1, \quad \frac{\sigma}{l^2} < 1 \tag{5.35}$$

In the same manner, substituting u from equation (5.34) in equation (5.33) yields

$$W = \frac{\rho}{2} \frac{l}{\sqrt{\sigma}} \frac{\Sigma^2}{\sigma^2} \varphi \left(\frac{\omega\sigma^{3/2}S}{\Sigma Q}\right) \frac{Q}{S} \frac{l}{S}$$

for

$$\frac{u\sigma}{\upsilon l} \gg 1 \qquad \frac{\sigma}{l^2} < 1 \tag{5.36}$$

where φ is the value of F_2 for small values of $\upsilon l/u\sigma$ and σ/l^2. If the frequencies of the pulsations are not large, W will increase linearly with an increase in Q. In the case W = 0, by the author's measurements (ref. 54), the numerical value of the coefficients ψ and φ is such that

$$W_0 = 2.5 \cdot 10^{-3} \frac{\Sigma}{\sigma^2} \frac{l}{S} \cdot \frac{g}{(\text{sec})(\text{cm})^4}$$

for

$$Q < 1.5 \cdot \frac{Sl}{\Sigma} \tag{5.37}$$

and

$$W = 2 \cdot 10^{-2} W_0 \frac{\Sigma}{\sqrt{\sigma}} \frac{Q}{S} \frac{g}{(\text{sec})(\text{cm})^4}$$

for

$$Q \gg 1.5 \frac{Sl}{\Sigma} \tag{5.38}$$

For the correct application of equations (5.37) and (5.38), it is necessary to take

$$\left.\begin{aligned} \sigma &= ab \\ \Sigma &= (a + d)(b + d) \\ l &= 2d \end{aligned}\right\} \tag{5.39}$$

where a and b are the lengths of the sides of the openings and d is the thickness of the fibers of the netting or fabric. It is necessary to bear in mind that the last equation is valid only for the condition $W \gg W_0$. In the intermediate region, the resistance must be considered as the sum $W' \cong (W + W_0)/2$. This intermediate region corresponds, as experience shows, to the values of the Reynolds number $u\sigma/\upsilon l \cong 10$, which corresponds to the values of the volume rate of flow Q indicated

in equations (5.37) and (5.38). Figure 50 shows the form of the resistance W as a function of Q for fabric with $a = b = 1\times10^{-2}$ centimeter and $d = 8\times10^{-2}$. The transition from the resistance W_0, independent of the velocity, to the resistance W proportional to the velocity, as predicted by theory, is apparent in figure 50. The computation of the resistance of a fabric or netting permits the evaluation of the velocity u of the flow of air through a meshed screen.[39] The value of the pressure drop of the air flowing through the meshed screen will be

$$\Delta p = \beta \frac{\rho v^2}{2} \tag{5.40}$$

where $\beta < 1$; this pressure drop is equal to

$$\beta\rho \frac{v^2}{2} = WQ = WS \frac{Q}{S} = wl \frac{Q}{S} \tag{5.41}$$

The magnitude Q/S is the mean velocity of the flow v'. For $u\sigma/\upsilon l \leq 10$, the unit resistance wl (it has the dimensions mechanical ohm/cm^2, and 1 mechanical ohm = 1 g/sec) is constant and equal to $w_0 l$. Hence from equation (5.41),

$$\frac{v'}{v} = \frac{\beta\rho v}{2w_0 l} \left(\text{for } \frac{u\sigma}{\upsilon l} < 10\right)$$

$$u = \frac{v'\Sigma}{\sigma} \tag{5.42}$$

From these equations for a fabric with $w_0 l$ = 10 mechanical ohm/cm^2 and v = 5 m/sec, $v'/v = 3\beta\times10^{-2} \ll 1$. The term $u\sigma/\upsilon l = v'\Sigma/\upsilon l = 7.5\cdot\beta \ll 10$ (assuming $\Sigma = 10^{-3}$ and $l = 2\cdot10^{-2}$). For larger values of $u\sigma/\upsilon l$, equations (5.38) and (5.41) yield

$$\frac{v'}{v} = \left(\frac{\beta\rho}{2\gamma}\right)^{1/2} \left(\text{for } \frac{v'\Sigma}{\upsilon l} \gg 10\right) \tag{5.43}$$

where γ is the coefficient of proportionality between the unit resistance wl and the velocity v':

$$wl = \gamma v' \tag{5.44}$$

[39] All these considerations refer also to the flow of water, but the numerical coefficients in equations (5.37) and (5.38) will be different. Further, the fabric will swell up in water so that its dimensions will change considerably.

On the basis of equation (5.38) $\gamma = 2 \cdot 10^{-2} w_0 l \Sigma / \sqrt{\sigma}$. For the same data and $\sigma = 10^{-4}$,

$$\gamma = 2 \cdot 10^{-2}$$

$$v'/v = 0.17\beta^{1/2}$$

and

$$v'\Sigma/\upsilon l = 8 \cdot 10^{-2} \beta^{1/2} v$$

so that for $v = 10$ meters per second, $v'\Sigma/\upsilon l = 80\beta^{1/2}$. In this case the velocity v' of the flow through the screen is linearly connected with the velocity of the approaching flow v. From equations (5.42) and (5.43) it is apparent that a considerable lowering of the velocity within the screen for moderate resistances (about 10 mechanical ohm/cm^4) of its fabric can be obtained. The extent such a screen will lower the intensity of the sound of the arriving useful signal has to be considered. Actually, the resistance in this case must be computed by equation (5.35) for $\omega \neq 0$. The magnitude $\psi(\omega\sigma/\upsilon)$, although it increases also in this case, nevertheless still remains a magnitude on the order of 1 (for medium frequencies and small values of σ). It is therefore possible to take the value of W for $\omega = 0$. If the resistivity is w, the pressure drop of the sound wave will be $dp = - w\, dx \cdot Q/S = - w\, dx\, \dot{\xi}$, where $\dot{\xi}$ is the velocity of the fluctuations. This magnitude is equal to $p/\rho c$ where c is the velocity of sound. Thus

$$dp = - \frac{w}{\rho c} p\, dx \tag{5.45}$$

therefore for the total thickness l of the screen,

$$p = p_0 e^{-\frac{wl}{\rho c}} \tag{5.46}$$

That is, the drop in intensity of the sound in decibels will be

$$I(db) = - 20 \log e \cdot \frac{wl}{\rho c} = - 0.20wl \tag{5.47}$$

that is, for example, for $wl = 10$ mechanical ohms per centimeter^4 only about 2 decibels. Thus, without conflicting with the sound transparency, it is possible to lower the velocity of the flow within the screen and thereby lower the frequencies and intensities of the vortices.

In connection with screens, it is of interest to point out another method of their application for eliminating the vortex formation. This elimination depends on the fact that a flow passing through a sufficiently transparent screen becomes turbulent, the frequencies of the vortices being then determined by the dimensions of the meshes of the screen δ and the velocity of the flow ($f''' = kv/\delta$). These frequencies may be so high that they appear beyond the limits of the frequency range of the receiver. The placing of such a screen near (or around) the receiver will not, of course, shield the receiver from the pressure pulsations in the flow if it is nonstationary, but the vortex formation on the body of the receiver will be artificially displaced toward the region of high frequencies f'''. This effect is shown, for example, by the protective screens that cover the mouths of loudspeakers (fig. 51). If there were no screen, the frequencies of the vortices would be determined by the dimensions of the opening of the loudspeaker. The screens displace the spectrum of the vortices toward the higher frequencies. This breaking down of the vortices is very well shown in the excitation of resonators by an air stream, discussed in section 27. If at the mouth of such an excited resonator a screen S'S" is placed intersecting the stream (fig. 52), the excitation of the resonator is immediatedly cut off, even for the case of a very rough screen (meshes of the order of 1 cm^2). This change in the scale of the vortices is also employed for the absorption of vibrations in open wind tunnels by placing near the opening of the tunnel from which the vortices are shed projecting lugs which break down these vortices into smaller ones (ref. 50).

30. Shielding of Sound Receiver from Velocity Pulsations of Approaching Flow

If the approaching flow is not steady, the problem arises of shielding the sound receiver from the pressure pulsations brought about by the nonsteady flow. In section 28 it was explained that, for the condition where the mean velocity of the flow v is much greater than the velocity pulsations δv and for the condition where the dimensions of the receiver d are much less than the dimensions of the pulsations Λ, the nonsteady flow about the body of the receiver may be considered on the basis of a knowledge of the flow picture for the steady flow. This permits making use of the important results from the theorem of Bernoulli applied to the flow about a body. In the flow about bodies, due to the compression of the stream, the velocity of the stream on the lateral sides of the body increases, while ahead of and behind the body it is slowed down. As a result, by the law of Bernoulli,

$$p = \text{constant} - \rho \frac{v^2}{2} \tag{5.48}$$

the pressure on the lateral surfaces of the body drops, while it increases ahead of and behind the body. On figure 53 is shown the pressure distribution over the surface of a sphere and of a streamlined body. A particularly interesting picture is evidenced in the case of the streamlined body.

The pressure in the middle part of the surface is not only negative but is very small in absolute value. Hence, if the sound receiver is placed in this part of the body, the change in pressure produced by the velocity pulsations of the flow may be very small. For a suitable choice of the shape of the body, the local coefficient β could be made to attain a value between the pressure p and the dynamic pressure $\rho v^2/2$ up to 0.02. We may note that for a mean velocity of the flow directed along the axis of the body $(d\beta/d\psi)_0 = 0$ so that equation (5.13) reduces to

$$p' = \beta_0 \rho v \, \delta v \tag{5.49}$$

The receiver diaphragm may be made flush with the surface of the deflector at the place where β_0 is minimum, or the receiver may be placed within the deflector, making a part of its screen transparent to sound. The sound-transparent surface must be very smooth and not too transparent to the flow; otherwise the flow about the body may change considerably.

The pressure distribution over the body with maximums ahead of and behind it and a minimum at the lateral sides suggests still another method for dealing with the pressure pulsations, namely, the principle of compensating the pressures. The essential character of this principle will be described in a simplified idealized form by imagining a body of the type illustrated in figure 53 placed in a stream. Inside the body there will be a chamber with a pressure receiver in it. A part of the surface of the deflector will be made transparent to the flow, for example, at the forward part where the pressure is positive and at the sides where it is negative (fig. 54). Under these conditions there will be a stream of air through the chamber. The velocity of this stream normal to the surface of the deflector will be denoted by v_n, and the difference in the pressures outside and inside the transparent partition by $\Delta p = p_a - p_i$. The flow of air passing through the area ds will then be equal to:

$$dL = v_n \cdot ds \qquad v_n = \frac{\Delta p}{W} = \frac{p_a - p_i}{W} \tag{5.50}$$

where W is the resistance of the partition. The total flow through the entire transparent parts of the surface of the deflector will be:

$$L = \int v_n ds = \int \frac{p_a - p_i}{W} \cdot ds \tag{5.51}$$

On account of the incompressibility of the fluid, this flow must be equal to zero; hence,

$$\int \frac{p_a \cdot ds}{W} = \int \frac{p_i \cdot ds}{W} \tag{5.52}$$

If W is constant, it follows that the mean outside pressure is equal to the mean inside pressure:

$$\overline{p}_a = \overline{p}_i \tag{5.53}$$

The mean is taken over the transparent parts of the deflector. If W is large, the interior velocities will be small and the pressure p_i may be assumed as practically constant over the entire volume so that $p_i = \overline{p}_i$. By choosing the position of the transparent surfaces and their resistance W and making use of the fact that at some places $p_a > 0$ and at others $p_a < 0$, $\overline{p}_a$ may be made equal to zero; from equation (5.33) it then follows that $\overline{p}_i = 0$. Thus, a chamber of constant pressure is derived. An example of this type of chamber is illustrated in figure 54. The parts of the deflector transparent to the air flow in the given case are located forward (screen s_1) and at the sides (screen s_2). At s_1, $p_a > 0$ and at s_2, $p < 0$. The third screen s_3 breaks up the additional stream entering the chamber through the opening s_1. In this case, it was found possible to attain the value $\beta_0 = 0.001$ for the position of the microphone M shown in the figure so that the pressure near M was only a thousandth of the dynamic pressure. The screens were still entirely transparent to the sound. The screens are important also from the viewpoint that, using acoustical terminology, they represent only active resistances since the enclosed chamber possesses no resonances. If for example, the deflector is made rigid with a small number of openings, a resonator of the Helmholtz type is obtained which strongly distorts the frequency characteristic of the receiver device.

The preceding discussed principle of pressure compensation leading to the formation of a constant-pressure chamber in many cases partially acts, so to speak, by itself. In fact, if the receiver is placed inside a deflector provided with walls transparent to the flow, it is sufficient that a part of the flow enter the chamber at $p_a > p_i$ and issue at $p_a < p_i$ for at least partial compensation to take place. Such partial compensation will be obtained, for example, in a screen deflector having

the shape of a sphere (see fig. 49) on the surface of which, for a sufficiently thick screen, there will occur both positive and negative pressures (see diagram of pressures in fig. 53).

As an example there may be cited the shield of a loudspeaker (fig. 55) made in the form of a sound-transparent sphere which encloses the opening of the speaker. If the transparency of such a sphere for the air flow is small, the pressures will be distributed as shown in the figure by the + and - signs, and compensation of the pressure pulsations due to the velocity pulsations will to a certain extent be obtained.

It must, however, be borne in mind that all the conclusions refer to that part of the pressure pulsations which is produced by a change in the dynamic pressure. The local changes of the velocity, as has been explained, likewise lead to pressure pulsations of the form $p'' = \rho a \frac{\partial v}{\partial t}$. This part of the pulsations remains even on the lowering of the magnitude $p' = \beta' \rho v^2/2$ so that it will form a certain background serving as the limit of the lowering of the acoustic interferences brought about by an unsteady flow. It is possible, of course, to suppose that this part of the pressure may likewise be subject to compensation, but for this there is no rational data because very little is known of the flow about bodies in a nonsteady stream. Moreover, the possibility of eliminating the interferences due to this part of the pressure may be doubted since they are essentially the same type as the pressure changes produced by a sound wave in a waveless region. The elimination of such interferences will therefore probably be in contradiction to the requirements of the sound receiver.

31. Sound Receiver Moving with Velocity

Considerably Less Than Velocity of Sound

The fundamental problem which is encountered in the mathematical theory of a moving receiver is that of computing the variable pressure produced by an approaching sound wave on the surface of a receiver, in particular its working part. This problem includes the computation of the flow about the body of the receiver, a computation associated with well-known difficulties.

A particularly difficult problem is that of the vortex formation arising behind the body. If the receiver body is of a well-streamlined shape, however, then, at least in its forward part, the flow may be considered as potential. If the working diaphragm is located in this part, the application of the potential-flow theory may be entirely practicable. In this section the idealized case of potential flow for $v \ll c$ will be considered.

In this case the equation for the potential of the sound wave φ, if terms of the order of v^2/c^2 are neglected, is (see eq. (1.85)):

$$\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = \Delta\varphi + 2\left(\frac{\nabla\Phi_0}{c^2}, \frac{\nabla\partial\varphi}{\partial t}\right) \tag{5.54}$$

where Φ_0 is the potential of the undisturbed flow about the body of the receiver ($\vec{v} = -\nabla\Phi_0$). This potential satisfies the equation

$$\Delta\Phi_0 = 0 \tag{5.55}$$

and the boundary condition

$$\frac{\partial\Phi_0}{\partial n} = 0 \quad \text{(on the surface of the body)} \tag{5.56}$$

where $\partial/\partial n$ is the derivative along the normal to the surface of the body. The required sound potential φ must satisfy equation (5.54) and the boundary condition

$$\frac{\partial\varphi}{\partial n} = 0 \quad \text{(on the surface of the body)} \tag{5.57}$$

(the yielding of the diaphragm is ignored). For a harmonic sound of frequency ω (the frequency ω is considered in a system of coordinates in which the receiver is stationary and the medium is in motion), set $\varphi = \varphi_0 e^{i\omega t}$. Then from equation (5.54)

$$\Delta\varphi_0 + k^2\varphi_0 + 2ik\frac{1}{c}(\nabla\Phi_0, \nabla\varphi_0) = 0 \qquad k = \frac{\omega}{c} \tag{5.54'}$$

The potential of the sound wave ψ in the absence of the flow is assumed known. This potential must then satisfy the equation

$$\Delta\psi_0 + k^2\psi_0 = 0 \tag{5.54''}$$

and the condition $\partial\psi_0/\partial n = 0$ on the surface of the receiver. Setting

$$\varphi_0 = \psi_0 \cdot e^{-i\frac{k\Phi_0}{c}} \tag{5.58}$$

where Φ_0 is the potential of the flow and then substituting this solution in equation (5.54') satisfy the equation (5.54") when terms of the order of v^2/c^2 are neglected. Equation (5.58) may therefore be considered as the solution of equation (5.54) with the assumed accuracy.

This solution satisfies also boundary condition (5.57). In fact,

$$\frac{\partial \varphi}{\partial n} = \frac{\partial \varphi_0}{\partial n} e^{i\omega t} = e^{i\left(\omega t - \frac{k\Phi_0}{c}\right)} \left(\frac{\partial \psi_0}{\partial n} - \frac{ik}{c} \frac{\partial \Phi_0}{\partial n} \psi_0\right) = 0 \qquad (5.57')$$

on the surface of the body, since $\partial\psi_0/\partial n$ and $\partial\Phi_0/\partial n$ are equal to zero. Thus, if the wave field ψ_0 is known near the stationary receiver, then with an accuracy up to v/c the field near the moving receiver φ_0 is obtained with the aid of the simple equation (5.58).

In particular, if there is considered a plane wave propagated, for example, in the oz-direction, the solution for the stationary receiver will be

$$\psi_0 = Ae^{-ikz} + S(r,\theta,\varphi) \qquad (5.59)$$

where Ae^{-ikz} is the incident wave and S is the dissipated wave. For large r, S must have the form $S = B(\theta,\varphi)e^{-ikr}/r$, where $B(\theta,\varphi)$ is the amplitude of the dissipated wave at a large distance from the body. It depends only on the angles θ and φ, which determines the direction of the dissipated beam. The solution for the moving receiver will be:

$$\varphi = A \,.\, e^{i\left[\omega t - kz - k\frac{\Phi_0}{c}\right]} + S(r,\theta,\varphi)e^{i\left[\omega t - k\frac{\Phi_0}{c}\right]} \qquad (5.60)$$

At a large distance from the body, the factor $e^{-ik\Phi_0/c}$ essentially gives the Doppler effect. In fact, for illustration take the case of a plane wave moving on the body with velocity v along the z-axis. Then at a large distance from the body $\Phi_0 = -vz$, and therefore the phase of the approaching wave is $[\omega t - k(1-v/c)z]$. Consider next a system of coordinates in which the flow is at rest, $\zeta = z - vt$. In this system the phase of the wave will be $[\omega t - k(1-v/c)z] = [\omega(1-v/c)t - k(1-v/c)\zeta]$, and therefore the frequency is equal to $\omega_0 = \omega(1-v/c)$. Hence the frequency in the system in which the body is at rest will be

$$\omega = \frac{\omega_0}{1 - v/c} = \omega_0 \left(1 + \frac{v}{c}\right) + \cdots \qquad (5.61)$$

as must be the case by the equation for the Doppler effect. The variable pressure on the surface of the receiver will now be computed. For this purpose use is made of the equation of Bernoulli, according to which

$$w = \int \frac{dp}{\rho} = \text{constant} + \frac{\partial \varphi}{\partial t} - \frac{[\nabla(\Phi_0 + \varphi)]^2}{2} \qquad (5.62)$$

Since the small change $\delta w = \int^{p+\pi} dp/\rho - \int^{p} dp/\rho = \pi/\rho$, where π is the sonic pressure, there is obtained for the variable part of the pressure

$$\frac{\pi_0}{\rho} = \left[i\omega\psi_0 - (\nabla\Phi_0, \nabla\psi_0) \right] e^{-i\frac{k\Phi_0}{c}} + \text{terms of higher order} \tag{5.63}$$

Far from the body where the flow becomes uniform, this equation does not give any results of interest. It confirms only the fact that the pressure π in a system of coordinates in which the body is at rest is the same as that in a system in which the body moves (for checking this statement, it is necessary to take into account the Doppler effect, by virtue of which, in a system of coordinates connected with the flow, $\partial\varphi/\partial t = \omega_0\varphi$ and not $\omega\varphi$).

Near the body the situation is otherwise. The magnitude ψ_0 near the surface of the body is of the order of magnitude equal to the amplitude of the incident wave A (see eq. (5.59)) and $\nabla\psi_0$ is of the order of magnitude equal to A/a, where a is the dimension of the body (here the tangential component of $\nabla\psi_0$ is considered; the normal component is equal to zero). Hence the first term in equation (5.63) is approximately equal to ωA and the second $\cong vA/a$. For $v/a > \omega$ the pressure on the surface of the receiver will be determined not by the first but by the second term. The amplitude of the potential A is connected with the amplitude of the pressure of the incident wave by the equation $A = \pi_0/i\rho\omega$. Hence, according to equation (5.63), the pressure on the surface of the receiver due to the first term on the right side of equation (5.63) will be $\pi' \cong \pi_0$ and that due to the second term will be $\pi'' = v\pi_0/a\omega$ and for $v/a > \omega$ may be greater than π'. That is, the characteristic amplification effect occurring in a moving receiver is obtained provided its dimensions are sufficiently small and the frequency of the sound is not too high. The condition of the presence of such amplification may, on the basis of what has been said, be written in the form

$$\frac{v}{c} > \frac{2\pi a}{\lambda} \tag{5.64}$$

where $v/c \ll 1$. The dimensions of the receiver must thus be very much smaller than the length of the sound wave λ.

32. Sound Receiver Moving with Velocity Exceeding Velocity of Sound

This case of the motion of a receiver presents special interest and at the same time special difficulties for theoretical computation. These difficulties are connected with the fact that to all the complexities of the problem of the flow about a body there is added the further feature of supersonic motion, the existence of density jumps (or shock waves) the occurrence of which was discussed in section 19. Instead of a solution of the problem posed, this section will be restricted, in addition to a few general remarks, to the discussion of the idealized simplest case which may serve as an orientation for a more detailed analysis of the problem of a receiver moving with supersonic velocity.

This problem has been the subject of frequent discussions (see, e.g., ref. 34) and various questions have been raised: Will the receiver in general receive the sound signal; will there exist a reflected wave; and so forth. There is, in fact, no basis for assuming that a receiver moving with supersonic velocity will not receive the variable pressure of a sound wave as soon as it is incident in its field. It is evident that it will always fall in its field provided the sound does not issue from a source located behind the receiver so that the sound is forced to overtake the receiver which for $v > c$ it cannot do.

The wave dissipated by the receiver will possess the characteristic that its entire field will lie behind the receiver in the Mach cone and moreover will be double (see section 20); that is, there will be two fields of different frequencies. For supersonic velocity of the receiver, however, the transmitted wave before reaching the receiver body must pass through the shock wave separating the part of the medium undisturbed by the motion of the body from the undisturbed part. Figure 56 illustrates what has been said for the case of a sound wave radiated by the source Q and received by the receiver P moving with velocity $v > c$. The curve M'MM" represents a section of the surface of the shock wave.

In connection with this, the question arises of the passing of a sound wave through a shock wave which is, in a way, a second screen of the receiver.

Under the usual conditions the presence of a sharp change of state of the medium would necessarily lead to the occurrence of two new waves, the reflected wave and the transmitted one. In this case, however, a reflected wave, as it is known, cannot be formed since the shock wave moves with supersonic velocity and without doubt would overtake the wave reflected from it. A certain light is thrown on this paradoxical

situation by the consideration of the simpler problem, namely, the transmission waves through a plane shock wave. It will be shown that in this case two transmitted waves arise of which one is a particular type.

Assume a straight density jump (shock wave) lying in a plane parallel to the plane $x = 0$ and moving in the direction of the positive x-axis (fig. 57) with velocity V. As was explained in section 19, the velocity V is greater than the velocity of sound in the medium at rest ($V > c_2$), in which the shock wave is displaced. The case will be considered in which a plane wave (from $x = + \infty$) is propagated so as to meet this shock wave. Since in the shock wave a jump in entropy occurs, recourse must be made to the general equations of the acoustics of a nonhomogeneous moving medium (eqs. (1.70), (1.71), (1.72) and (1.73)) if the propagation of sound is considered under these conditions. These equations for the one-dimensional problem which is being considered are

$$\frac{\partial \xi}{\partial t} + v \frac{\partial \xi}{\partial x} = - \frac{1}{\rho} \frac{\partial \pi}{\partial x} \qquad \frac{\partial \pi}{\partial x} = \left(\frac{\partial p}{\partial \rho}\right)_s \frac{\partial \delta}{\partial x} + \left(\frac{\partial p}{\partial s}\right)_\rho \frac{\partial \sigma}{\partial x} \tag{5.65}$$

$$\frac{\partial \delta}{\partial t} + v \frac{\partial \delta}{\partial x} + \rho \frac{\partial \xi}{\partial x} = 0 \tag{5.66}$$

$$\frac{\partial \sigma}{\partial t} + v \frac{\partial \sigma}{\partial x} = 0 \tag{5.67}$$

In these equation ξ is the velocity component of the sound vibrations along the x-axis ($\xi_y = \xi_z = 0$); v is the velocity of the medium along the x-axis ($v_y = v_z = 0$); δ, π, and σ are the changes in density of the gas, its pressure, and entropy, respectively, produced by the sound wave. The terms ∇p, $\nabla \rho$, and ∇s are neglected because p, ρ, and s are assumed constant on each side of the shock wave. If the entropy of the medium were everywhere constant, then, as was shown earlier (see section 4), $\sigma = 0$. In a shock wave, however, the entropy itself changes discontinuously so that it must not be assumed that s = constant and it is not legitimate to assume $\sigma = 0$ for the entire medium. In the incident wave, of course, propagated in a medium at rest (eq. (5.66)), $\sigma = 0$, since this wave may be considered as a usual adiabatic sound wave. With regard to the secondary waves arising as a result of the interaction of the incident sound wave with the shock wave, the question whether these waves are accompanied by changes in enthropy or not can be decided only on the basis of the consideration of the boundary conditions.

For the solution of the previously mentioned problem, it is convenient to use a system of coordinates in which the shock wave is at rest ($x' = x - Vt$). In this system the velocity of the medium is

$$u = v - V \tag{5.68}$$

and equations (5.65), (5.66), and (5.67) become

$$\frac{\partial \xi}{\partial t} + u \frac{\partial \xi}{\partial x'} = - \left(\frac{c^2}{\rho} \frac{\partial \delta}{\partial x'} + \frac{h}{\rho} \frac{\partial \sigma}{\partial x'} \right) \tag{5.65'}$$

$$\frac{\partial \delta}{\partial t} + u \frac{\partial \delta}{\partial x'} + \rho \frac{\partial \xi}{\partial x'} = 0 \tag{5.66'}$$

$$\frac{\partial \delta}{\partial t} + u \frac{\partial \sigma}{\partial x'} = 0 \tag{5.67'}$$

where in place of $(\partial p/\partial \rho)_s$ and $(\partial p/\partial s)_\rho$ are used the values

$$\left(\frac{\partial p}{\partial \rho}\right)_s = c^2 = \gamma \frac{p}{\rho}$$

$$\left(\frac{\partial p}{\partial s}\right)_\rho = h = \frac{p}{c_v} \tag{5.69}$$

With the assumption that the incident wave is a harmonic wave, ξ, δ, and σ may be set proportional to $e_i^{(\omega' t + k' x')}$, where ω' is the frequency (in the system x') and k' is the wave number. For such a wave, carrying out the differentiation in equations (5.65'), (5.66'), and (5.67'), yields

$$(\omega' + uk')\xi = - \frac{k'c^2}{\rho} \delta - \frac{h}{\rho} k'\sigma \tag{5.70}$$

$$(\omega' + uk')\delta = - \rho k' \xi \tag{5.71}$$

$$(\omega' + uk')\sigma = 0 \tag{5.72}$$

From the preceding two solutions are obtained: either $\sigma = 0$ or $(\omega' + uk') = 0$. In the first case, from equations (5.70) and (5.71) there is obtained

$$k' = \frac{\omega'}{\pm c - u}$$

$$\xi = \mp \frac{c\delta}{\rho}$$

$$\sigma = 0 \tag{5.73}$$

This solution represents the usual adiabatic sound wave, the phase velocity of which is equal to $V_f = \pm c - u$, as should be the case for a moving medium (if a system of coordinates is used in which the medium is at rest, i.e., $u = 0$, the $V_f = \pm c$), where c is the adiabatic velocity of sound in the medium under consideration.

The second solution of equations (5.70), (5.71), and (5.72) reads

$$k' = -\frac{\omega'}{u}$$

$$\xi = 0$$

$$\sigma = -\frac{c^2}{h}\delta \tag{5.74}$$

In this wave the velocity of the sound vibrations is equal to zero; while changes occur in the entropy σ and in the density δ of the medium, this wave does not, however, give rise to changes in pressure in the medium. In fact, $\pi = c^2\delta + h\sigma$. From equation (5.74) it follows that

$$\pi = 0 \tag{5.74'}$$

This conclusion is evident also from the fact that for the wave under consideration the velocity $\xi = 0$ so that the moving force must likewise be equal to zero. It is convenient to call this wave an entropy wave. As is seen from equation (5.74), this wave is propagated with a velocity equal to the velocity of motion of the medium u; that is, it is essentially simply carried along by the medium. Thus there are two types of wave. At first glance it appears that the discussion could be restricted to the usual isentropic waves (eq. (5.73)) and, with the possession of two independent solutions, the boundary conditions on the shock wave could be satisfied.

It could, in fact, easily be shown that from these solutions it is not possible to construct a solution satisfying the initial data which represent, for example, a restricted train of waves encountering the shock wave. On the contrary, in the problem presented herein, there must be unavoidably recourse to the solution (5.74); that is, in the shock wave there occur, as was already mentioned, irreversible

processes, and the disturbances of this shock wave will give rise to entropy fluctuations which will be propagated as a wave of the form of equation (5.74).

All these results are obtained automatically if the conditions on the surface of the discontinuity (see section 19) are used with the substitution of the differential equations of the hydrodynamics of a compressible fluid. According to equations (3.93), (3.94), and (3.95), these conditions are

$$
\begin{gathered}
u_1\rho_1 = u_2\rho_2 \\
\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 \\
w_1 + \frac{u_1^2}{2} = w_2 + \frac{u_2^2}{2}
\end{gathered}
\tag{5.75}
$$

where the subscripts 1 and 2 refer to the medium behind the shock wave (1) and to the medium at rest ahead of the shock wave (2). The first of these conditions expresses the law of the conservation of matter; the second, of momentum; and the third, of energy (w is the heat function). In the transmission of a sound wave, these conditions change since all the magnitudes receive small increments (ξ, δ, π, and σ). It must also be taken into account that $u = v - V$ and that the velocity of motion of the shock wave V must likewise be varied. The change in this velocity will be denoted by Δ. With the use of only linear approximation, the varied conditions which will be the boundary conditions for the sound wave on the shock wave will be obtained from equation (5.75). Thus, replacing in equation (5.75) u by $\xi - \Delta$, ρ by $\rho + \delta$, and p by $p + \pi$ yields

$$
u_1\delta_1 + (\xi_1 - \Delta)\rho_1 = u_2\delta_2 + (\xi_2 - \Delta)\rho_2 \tag{5.76}
$$

$$
u_1^2\delta_1 + 2(\xi_1 - \Delta)u_1\rho_1 + c_1^2\delta_1 + h_1\sigma_1 = u_2^2\delta_2 + 2(\xi_2 - \Delta)u_2\rho_2 + c_2^2\delta_2 + h_2\sigma_2 \tag{5.76'}
$$

(since $\pi = c^2\delta + h\sigma$) and finally from the third equation of equations (5.74), it is borne in mind that $\delta w = c^2\delta/\rho + c^2\sigma/r$ (r is the gas constant, $p = r\rho T$) there is obtained

$$
\frac{c_1^2}{\rho_1}\delta_1 + \frac{c_1^2}{r}\sigma_1 + u_1(\xi_1 - \Delta) = \frac{c_2^2}{\rho_2}\delta_2 + \frac{c_2^2}{r}\sigma_2 + u_2(\xi_2 - \Delta) \tag{5.76''}
$$

In the undisturbed medium ($x > 0$, subscript 2) there is only the incident wave. For this wave $\sigma_2 = 0$ and $\xi_2 = -c_2\delta_2/\rho_2$, where c_2 is the adiabatic velocity of sound in the undisturbed medium. In the medium behind the shock wave ($x < 0$, subscript 1) the change in density is $\delta_1 = \delta_1' + \delta_2''$, where δ_1' belongs to the transmitted isentropic sound wave for which $\xi_1' = -c_1\delta_1/\rho_2$ and δ_1'' belongs to the entropy wave for which $\xi_1'' = 0$ and $\sigma_1 = -c_1^2\delta_1''/h = -\gamma/(\gamma - 1) \times r\delta_1''/\rho_1$. With the use of these relations, ξ and σ are eliminated from equations (5.76), (5.76'), and (5.76"), and after simple algebraic transformations there are obtained

$$(u_1 - c_1)^2 \cdot \delta_1' + u_1^2 + u_1^2\delta_1'' = (u_2 - c_2)^2\delta_2 \tag{5.77}$$

$$\left[\frac{u_1^2}{\rho_2} + \frac{c_1^2}{\rho_1} - c_1u_1\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right]\delta_1' + \left[\frac{u_1^2}{\rho_2} - \frac{1}{\gamma - 1}\frac{c_1^2}{\rho_1}\right]\delta_2''$$

$$= \left[\frac{u_2^2}{\rho_1} + \frac{c_2^2}{\rho_2} - c_2u_2\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right]\delta_2 \tag{5.77'}$$

From conditions (5.75) it is possible to express the magnitudes characterizing the state of the gas behind the shock wave in terms of the ratio of pressures p_1/p_2, in the shock wave, and ahead of it (see section 19); there are then obtained

$$\frac{\rho_1}{\rho_2} = \frac{(\gamma - 1) + (\gamma + 1)\dfrac{p_1}{p_2}}{(\gamma + 1) + (\gamma - 1)\dfrac{p_1}{p_2}}$$

$$c_1^2 = c_2^2 \cdot \frac{p_1}{p_2}\,\frac{(\gamma - 1)\dfrac{p_1}{p_2} + (\gamma + 1)}{(\gamma + 1)\dfrac{p_1}{p_2} + (\gamma - 1)} \tag{5.78}$$

$$u_1^2 = \frac{c_2^2}{2\gamma}\,\frac{\left[(\gamma - 1)\dfrac{p_1}{p_2} + (\gamma + 1)\right]^2}{(\gamma + 1)\dfrac{p_1}{p_2} + (\gamma - 1)}$$

$$u_2^2 = \frac{c_2^2}{2\gamma}\left[(\gamma + 1)\frac{p_1}{p_2} + (\gamma - 1)\right] \tag{5.79}$$

where $u_1, u_2 < 0$, and $c_2^2 = \gamma p_2/\rho_2 = c^2$ is the square of the adiabatic velocity of sound in the undisturbed medium. The medium ahead of the shock wave will be assumed at rest so that $u_2 = v_2 - V = - V(v_2 = 0)$.

The magnitude δ_2 is given by the amplitude of the incident wave. Hence, from equations (5.78) and (5.79) it is possible to find δ_1' and δ_1'' for the transmitted waves. With these values it is possible, according to equations (5.73) and (5.74), to obtain the remaining characteristics of the transmitted waves. The pressure in the incident wave will be denoted by $\pi_0 = c^2\delta_2$ and the pressure in the transmitted acoustic wave, by $\pi' = c_1^2\delta_1'$ (since $\sigma_1 = 0$). With the elimination of δ_1'' from equations (5.77) and (5.77'), the ratio $\pi'/\pi_0 = c_1^2\delta_1'/c^2\delta_2$ is obtained:

$$\frac{\pi'}{\pi_0} = \frac{\left(\frac{u_2}{c_2} - 1\right)^2 \left[\frac{u_1^2}{c_1^2}\frac{1}{\rho_2} - \frac{1}{\gamma - 1}\cdot\frac{1}{\rho_1}\right] - \frac{u_1^2}{c_1^2}\left[\frac{u_2^2}{c_2^2}\frac{1}{\rho_1} + \frac{1}{\rho_2} - \frac{u_2}{c_2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right]}{\left(\frac{u_1}{c_1} - 1\right)^2 \left[\frac{u_1^2}{c_1^2}\frac{1}{\rho_2} - \frac{1}{\gamma - 1}\cdot\frac{1}{\rho_1}\right] - \frac{u_1^2}{c_1^2}\left[\frac{u_1^2}{c_1^2}\frac{1}{\rho_2} + \frac{1}{\rho_1} - \frac{u_1}{c_1}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\right]} \tag{5.80}$$

This ratio for small shock waves approaches 1, as should be the case, and is equal to

$$\frac{\pi'}{\pi_0} = 1 + \frac{5}{8}\cdot\frac{\gamma + 1}{\gamma}\cdot\frac{p_1 - p_2}{p_2} + \ldots \tag{5.81}$$

(for $(p_1 - p_2)/p_2 \ll 1$). For large shock waves there is obtained

$$\frac{\pi'}{\pi_0} = \frac{1}{\gamma}\cdot\frac{1}{1 + 2\left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}}\cdot\frac{p_1 - p_2}{p_2} + \ldots \tag{5.82}$$

(here $(p_1 - p_2)/p_2 \gg 1$). In both cases, the sound pressure in the transmitted wave increases in comparison with the pressure in the incident wave. The pressure of the entropy wave, as has already been pointed out, is equal to zero. Hence it is of less interest to find δ_1'' and σ_1 then it is to find the changes in temperature which occur because of the passage of the entropy wave. From the identity

$$p(\rho,T) = p(\rho,s) \tag{5.83}$$

there is obtained

$$\left(\frac{\partial p}{\partial \rho}\right)_T \delta + \left(\frac{\partial p}{\partial T}\right)_\rho \theta = \left(\frac{\partial p}{\partial \rho}\right)_s \delta + \left(\frac{\partial p}{\partial s}\right)_\rho \sigma \tag{5.84}$$

With the knowledge that $(\partial p/\partial \rho)_T = p/\rho = a^2$, where a is the isothermal velocity of sound and $(\partial p/\partial T)_\rho = p/T$, and with the fact that for the entropy wave $(\partial p/\partial \rho)_s \delta_1'' + (\partial p/\partial s)_\rho \sigma_1 = 0$, the temperature fluctuations produced by the sound wave are obtained in the form

$$\theta_1' = T_1(\gamma - 1) \cdot \frac{\delta_1'}{\rho_1} \tag{5.85}$$

and by the entropy wave

$$\theta_1'' = - T_1 \cdot \frac{\delta_1''}{\rho} \tag{5.86}$$

From the preceding it is seen that the role of both waves will be comparable only in the case where δ_1' and δ_1'' are of the same order. If from equations (5.77) and (5.77') δ_1'/δ_2 is eliminated, δ_1''/δ_2 is obtained. For small shock waves $(p_1 - p_2)/p_2 << 1)$ there is obtained

$$\frac{\delta_1''}{\delta_2} = \frac{\gamma - 1}{2\gamma} \cdot \frac{p_1 - p_2}{p_2} + \ldots \tag{5.87}$$

and since in this case δ_1'/δ_2 is near 1, the entropy wave does not play a marked role. Farther on its value increases, and with increase in the shock wave δ_1''/δ_2 approaches

$$\frac{\delta_1''}{\delta_2} = \frac{1}{\gamma} \cdot \frac{(\gamma - 1) + \left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}}{1 + 2\left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}} \tag{5.87'}$$

For the sound wave $\delta_1'/\delta_2 = \pi'/\pi_0 \cdot c^2/c_1^2$. As a result of equations (5.78) and (5.82), for large shock waves this ratio is equal to

$$\frac{\delta_1'}{\delta_2} = \frac{1}{\gamma} \frac{1}{1 + 2\left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}} \tag{5.87''}$$

Thus for $(p_1 - p_2)/p_2 >> 1$ the value of both waves in relation to the fluctuations of the temperature of the medium behind the shock wave becomes of the same order:

$$\theta_1' = T_1 \frac{\gamma - 1}{\gamma} \frac{1}{1 + 2\left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}} \cdot \frac{\delta_2}{\rho_1} \tag{5.88}$$

$$\theta_1'' = - T_1 \frac{1}{\gamma} \frac{(\gamma - 1) + \left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}}{1 + 2\left(\frac{\gamma - 1}{2\gamma}\right)^{1/2}} \frac{\delta_2}{\rho_1} \tag{5.88'}$$

From equation (5.82) it follows that for large shock waves the sound pressure behind the shock wave is intensified. From this, of course, there is not to be drawn any final conclusion as to the pressure on the sound receiver itself. It may be assumed that in the case of supersonic velocity of motion of the receiver there will also occur the intensification which was considered in section 29, based on the theorem of Bernoulli. This side of the question is not possible to analyze in greater detail because the supersonic flow about a body presents, as yet, a far from solved problem.

The very simple case considered herein leads to an explanation of the absence of a reflected wave when a sound wave passes through a shock wave, and there is no basis for thinking that this aspect of the matter would be subject to essential modification for shock waves of more complicated form (of the type illustrated in fig. 57).

A similar remark may be made on the existence of two waves behind the shock wave: the sound and entropy waves. With regard to the quantitative relations, the fact that for small shock waves the transmitted wave is almost undisturbed by the shock wave should likewise not depend on the shape of the shock wave and probably has a more general significance than follows directly from the special case considered.

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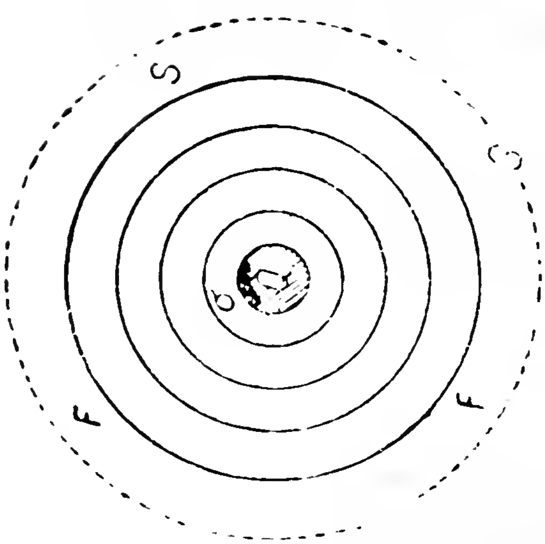

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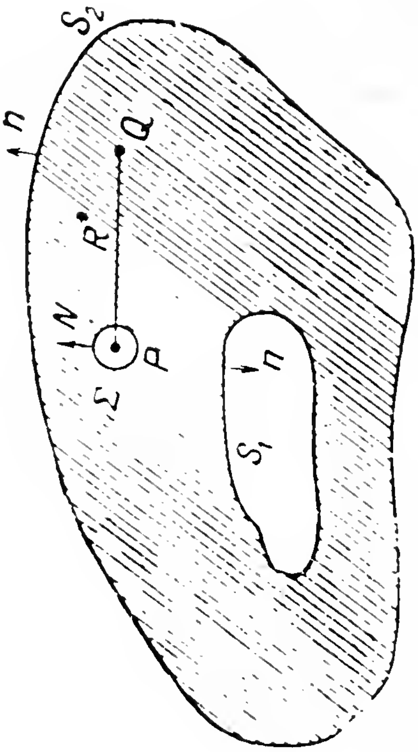

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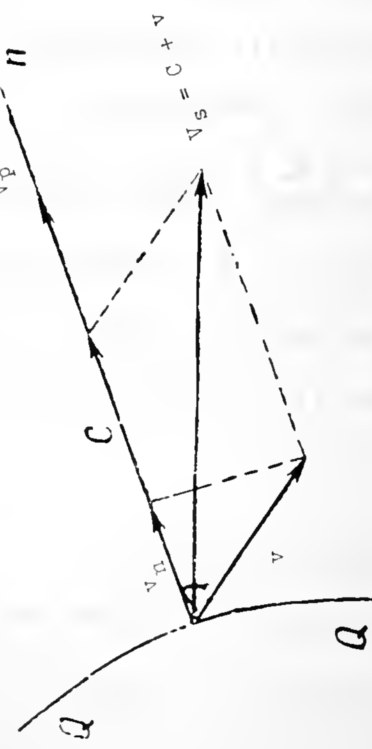

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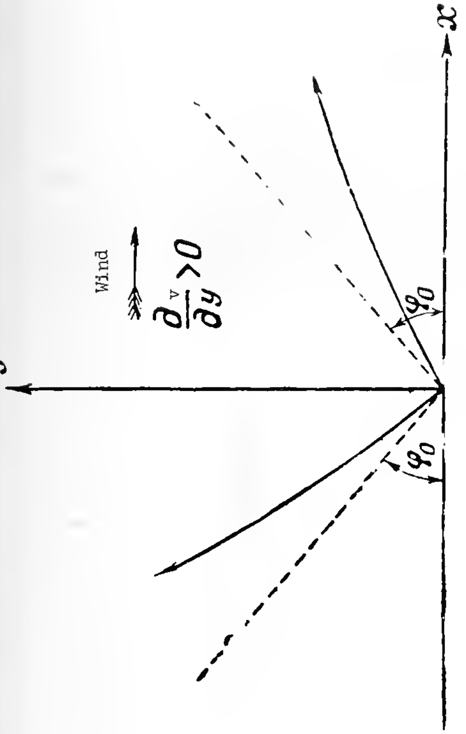

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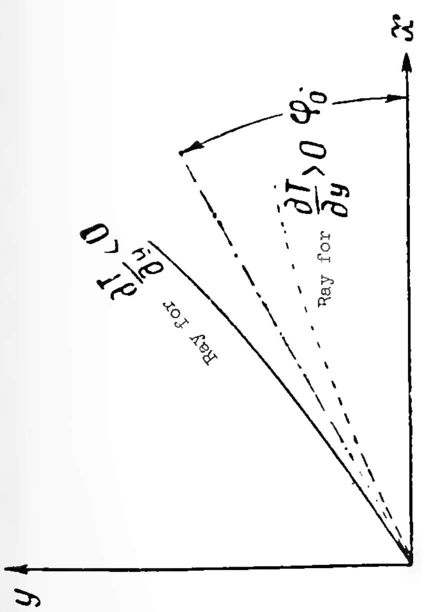

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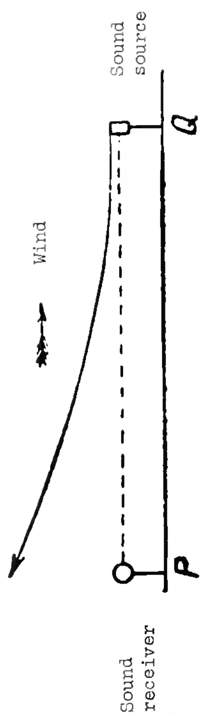

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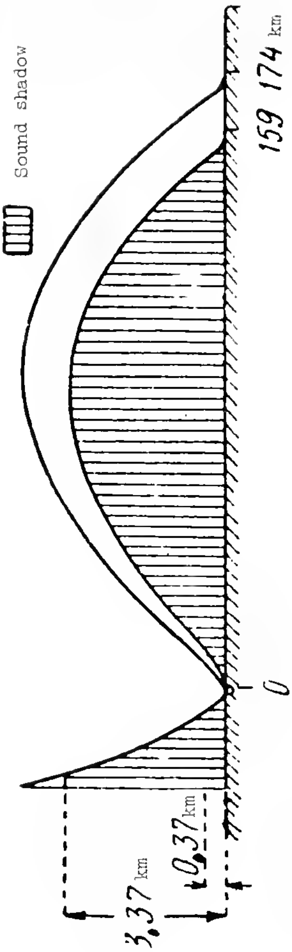

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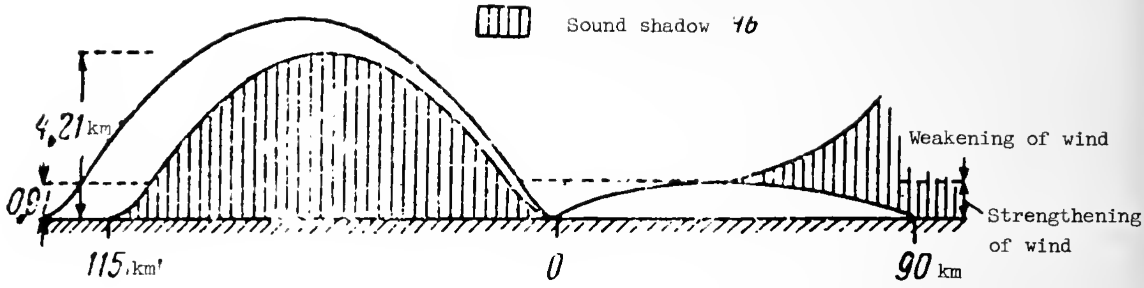

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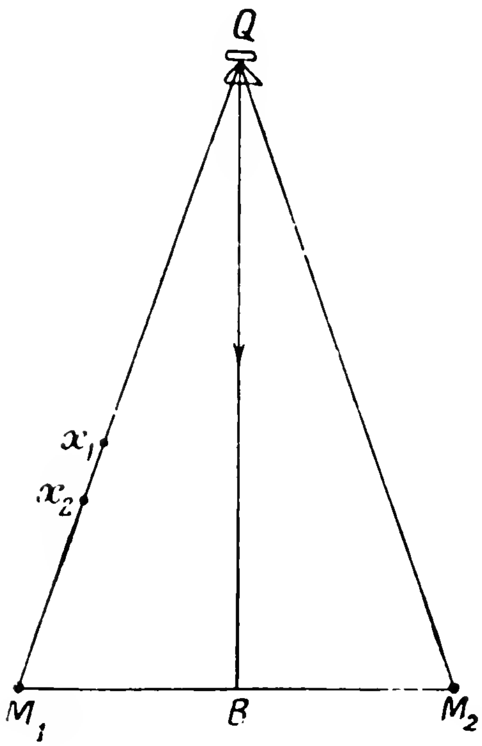

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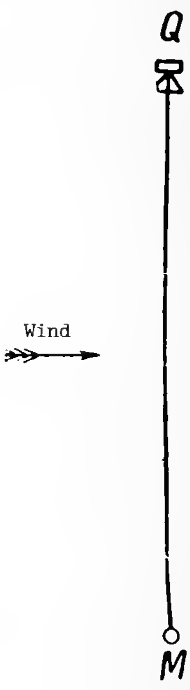

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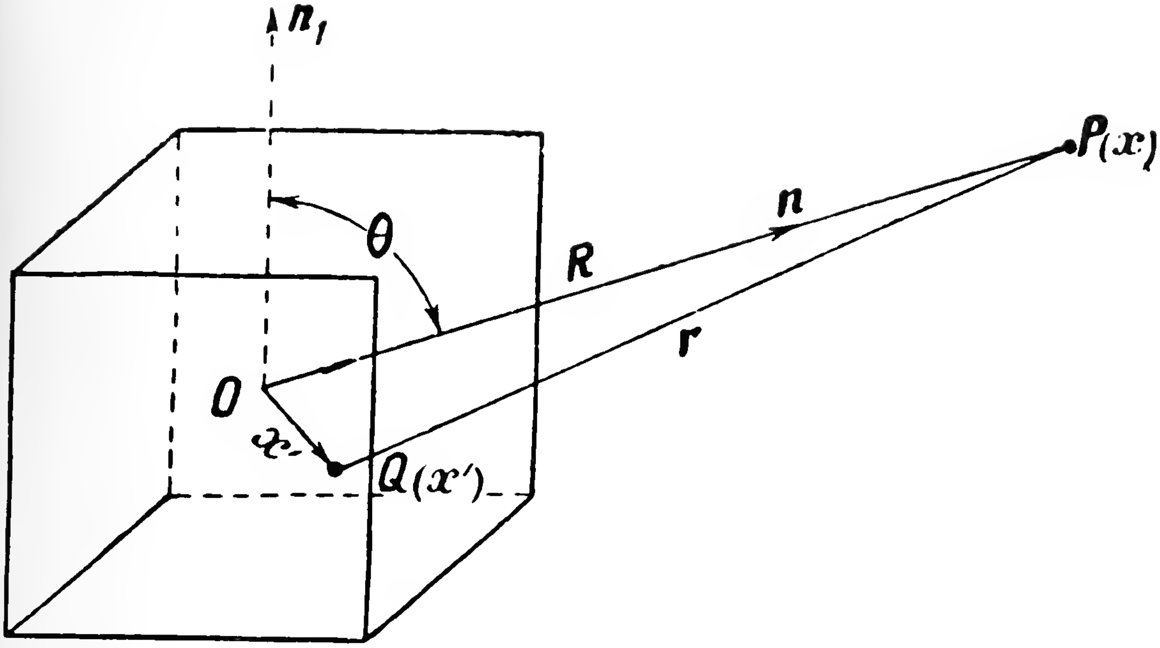

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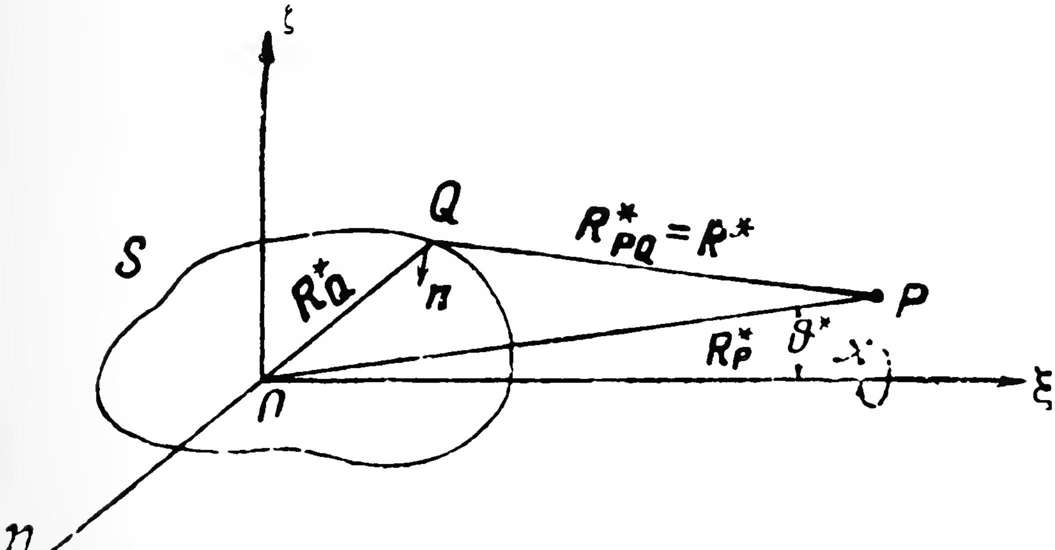

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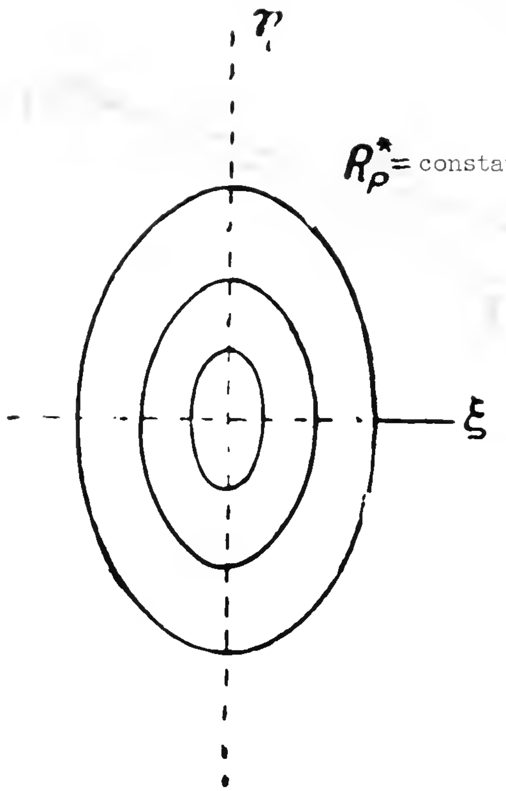

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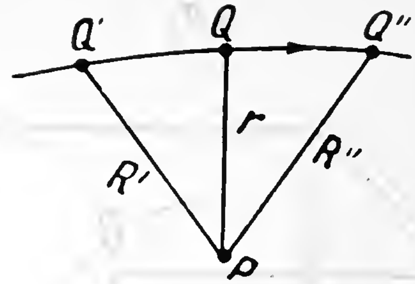

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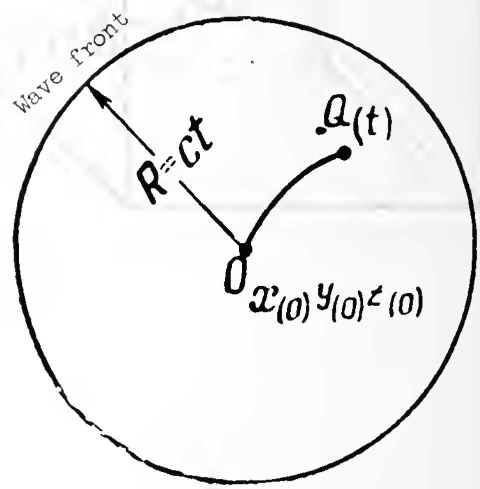

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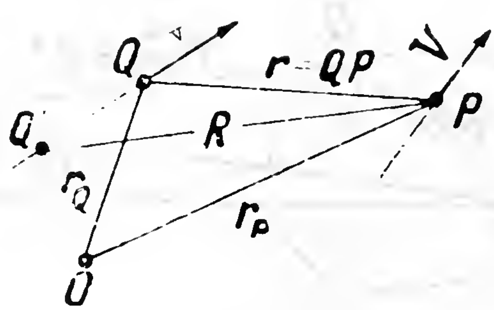

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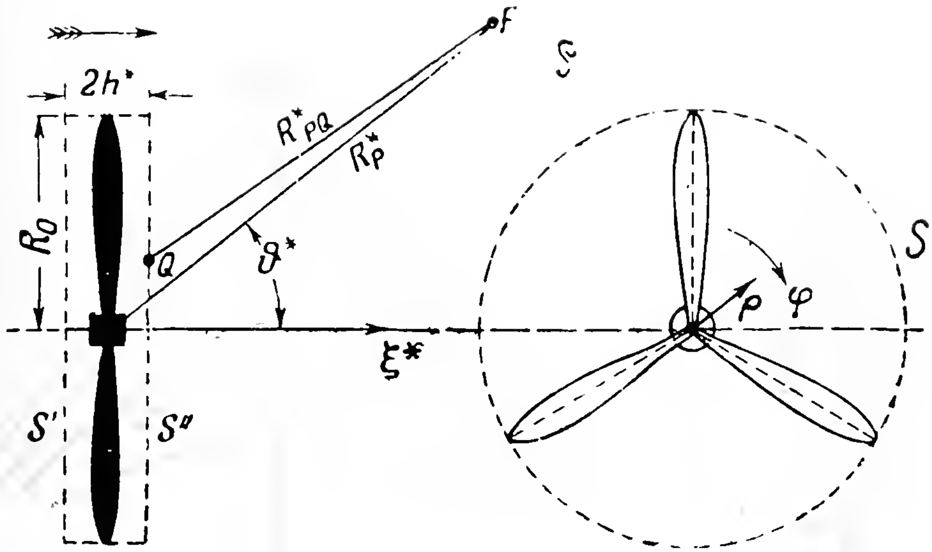

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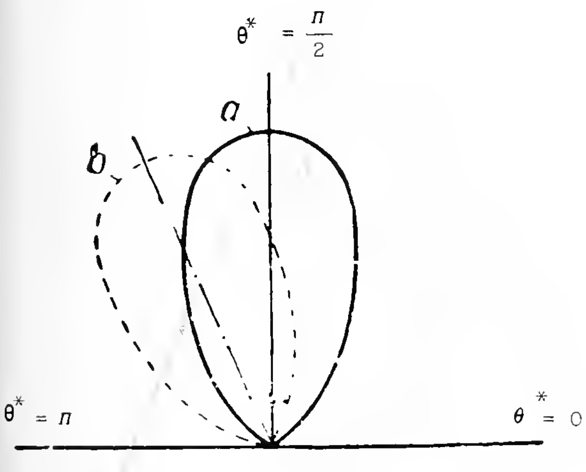

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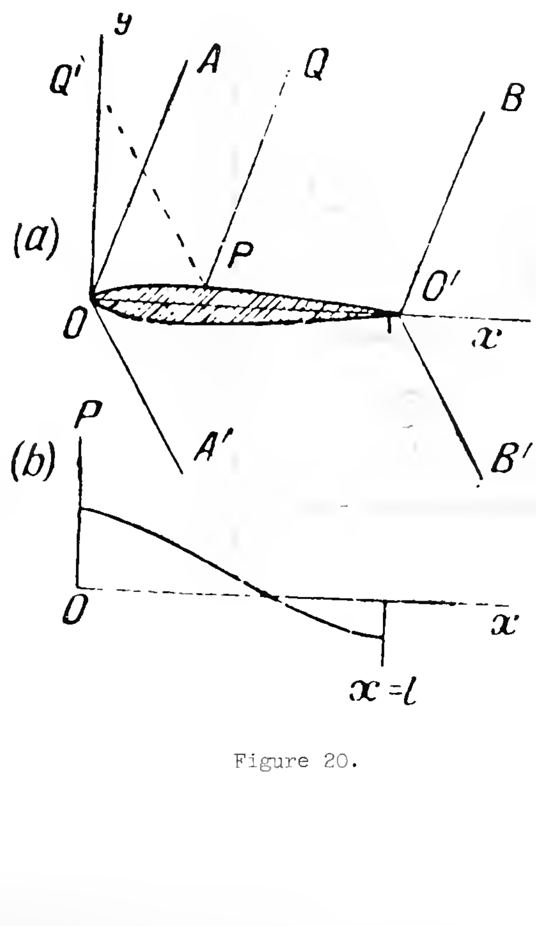

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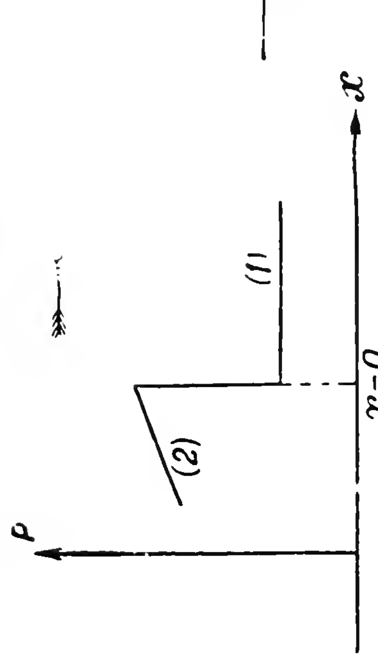

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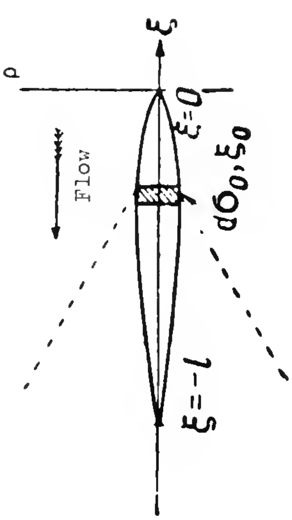

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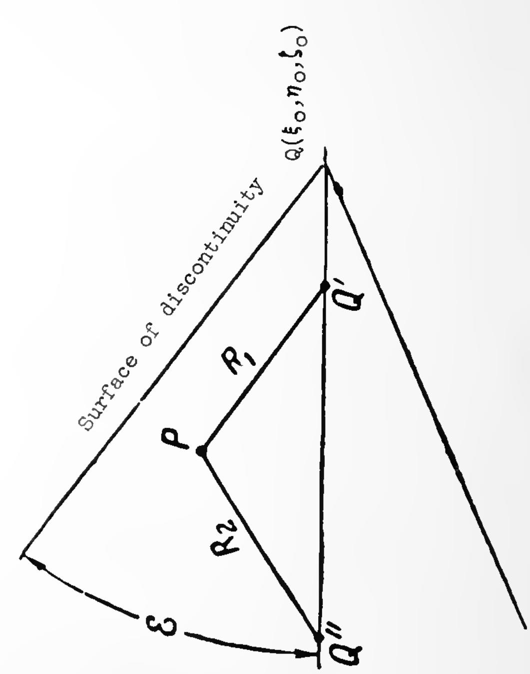

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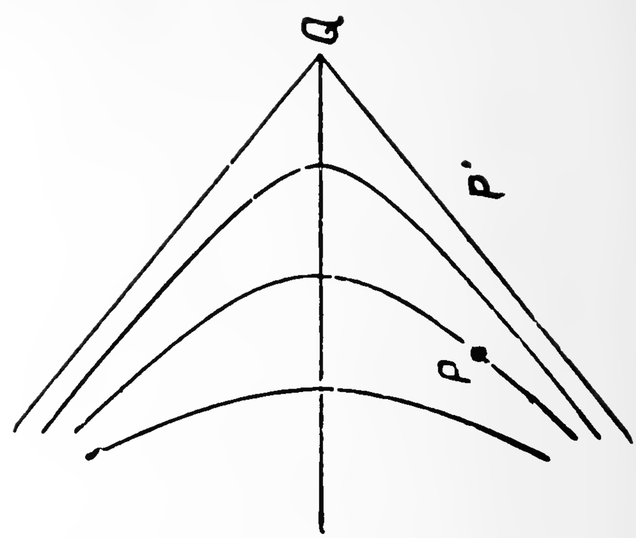

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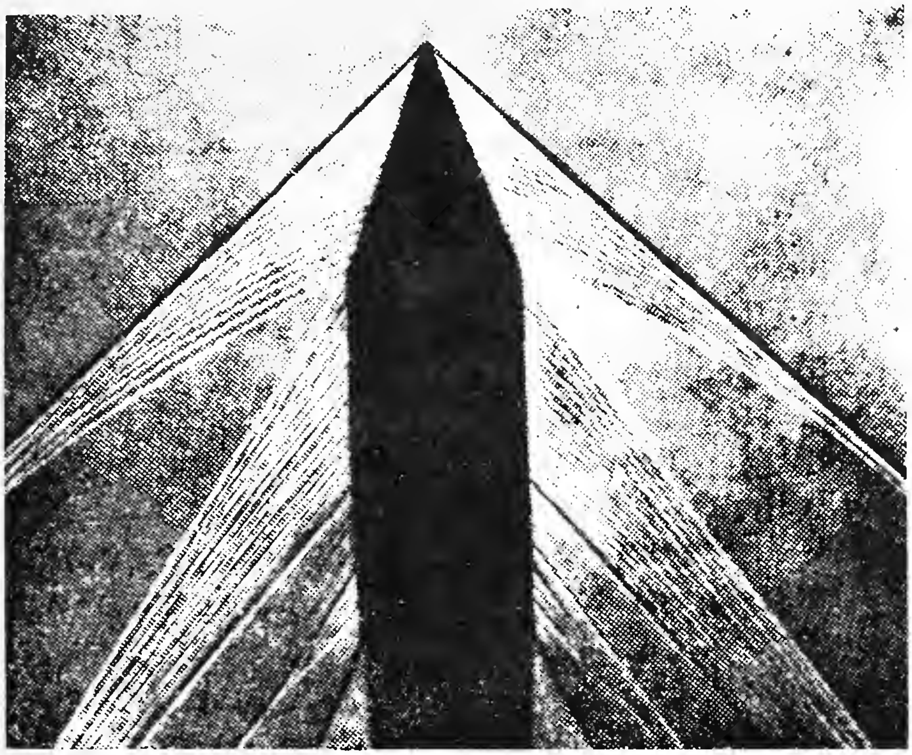

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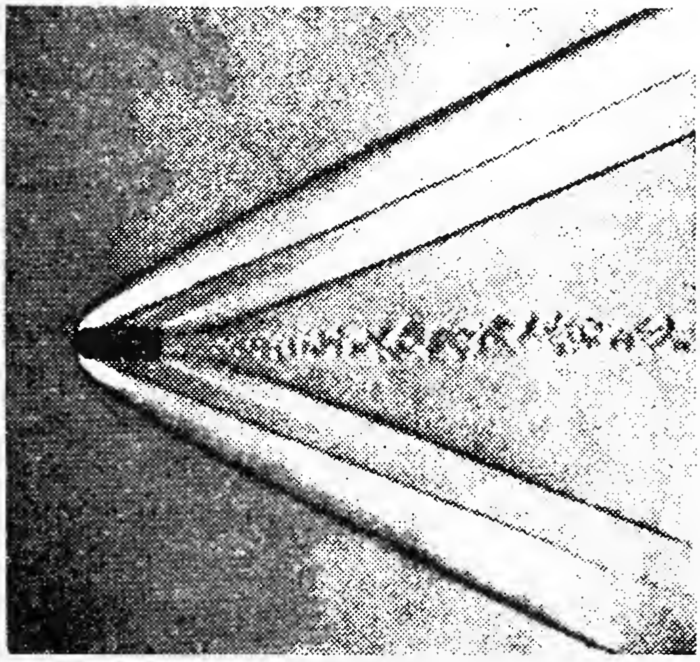

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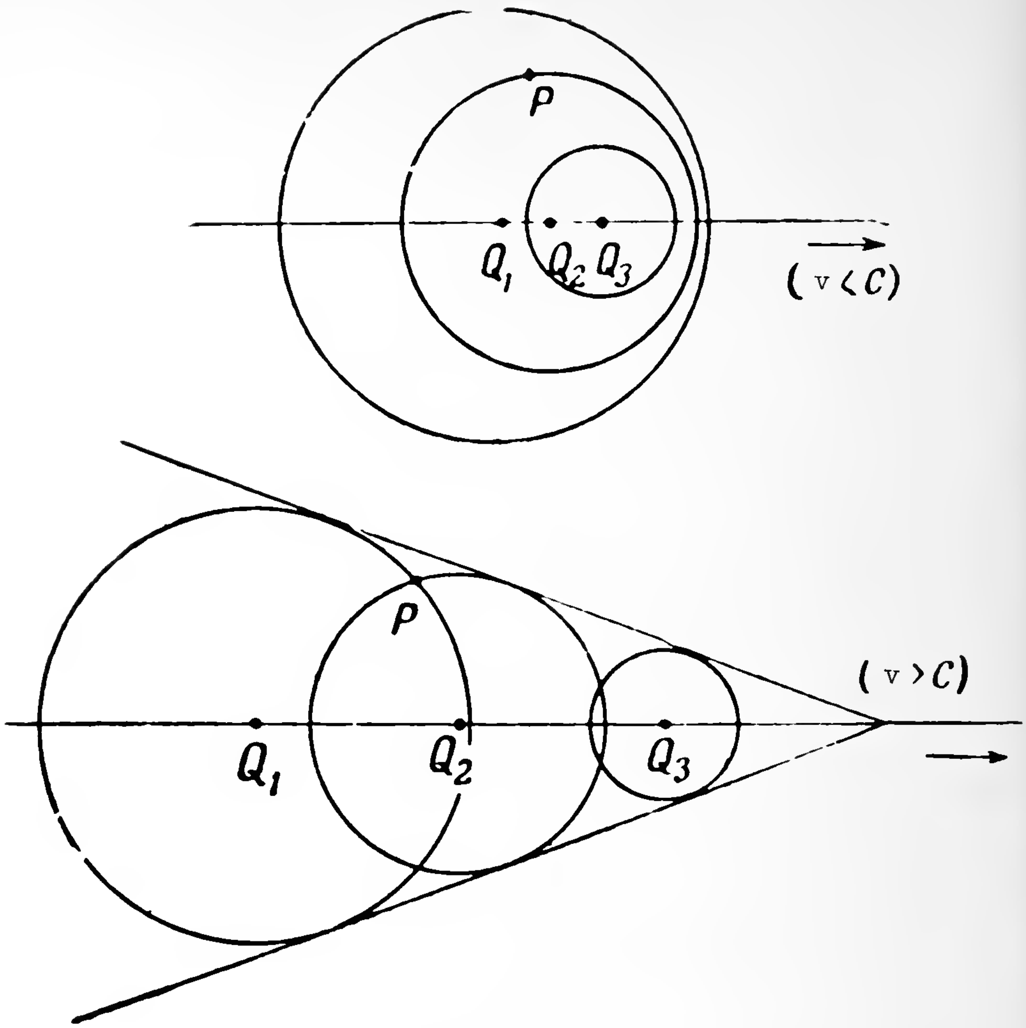

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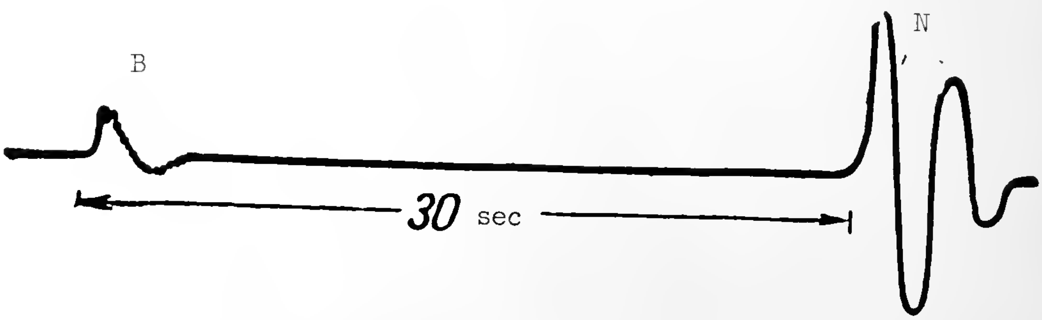

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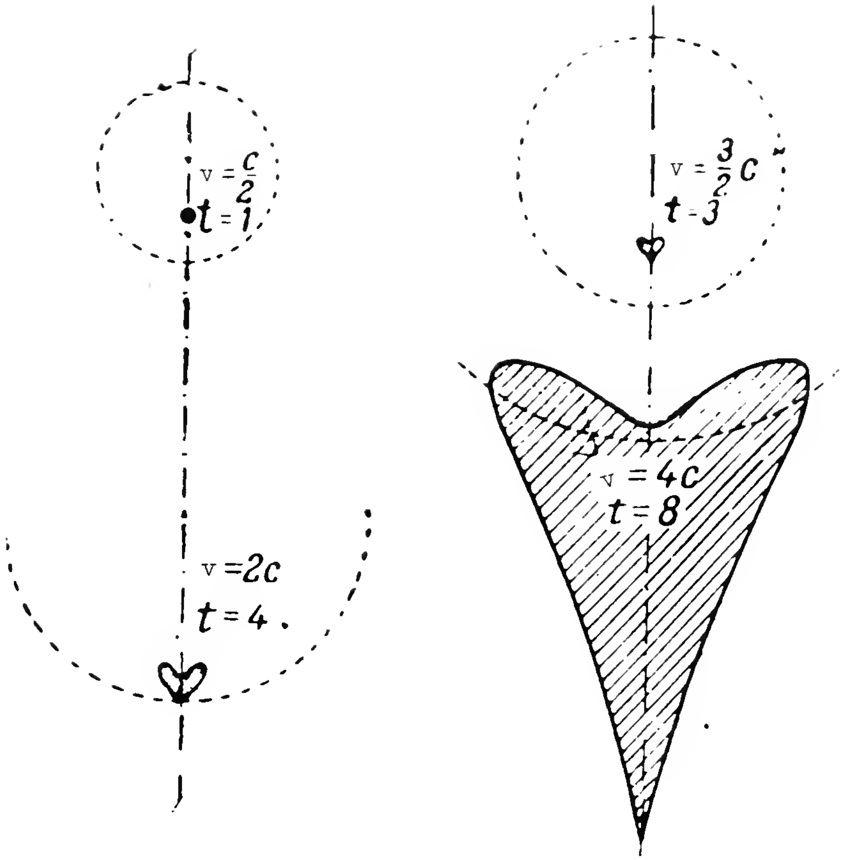

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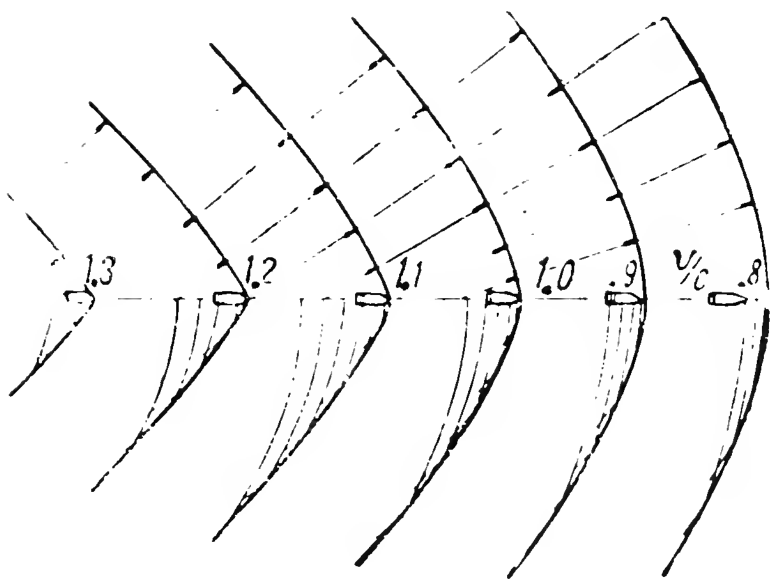

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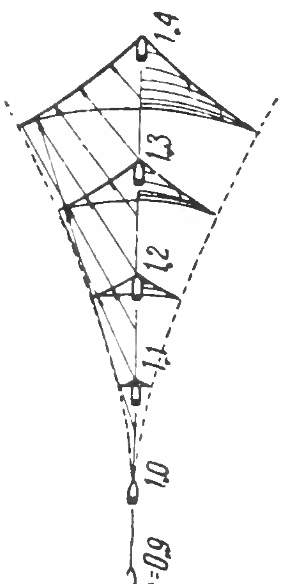

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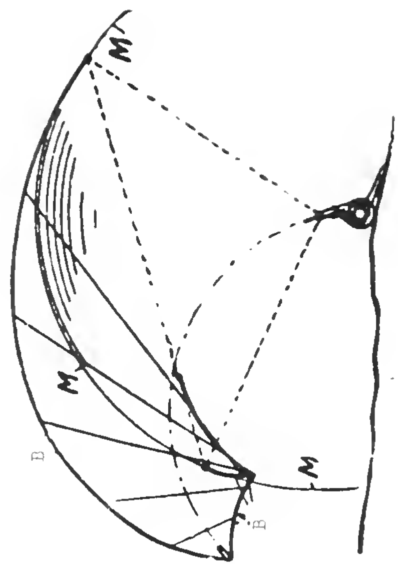

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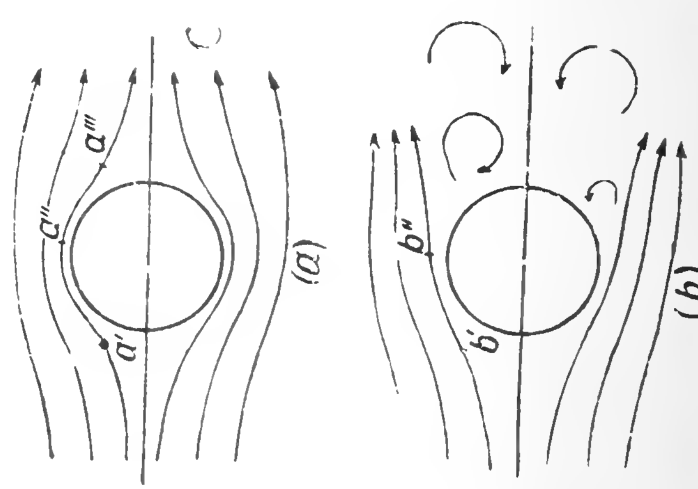

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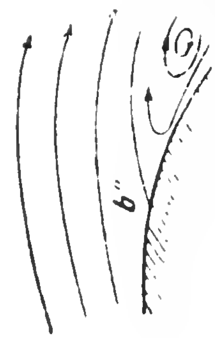

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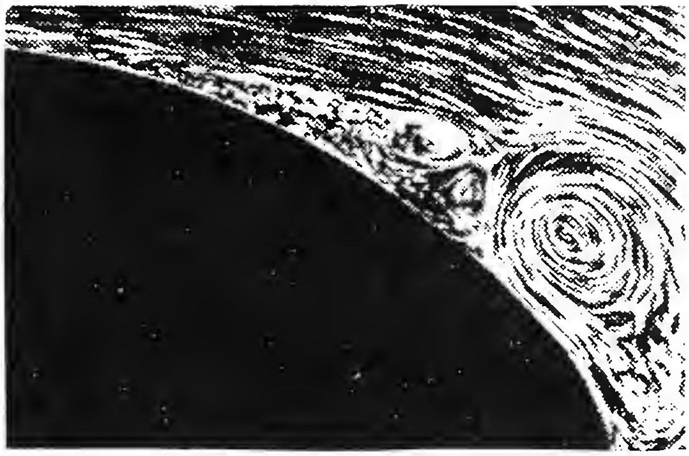

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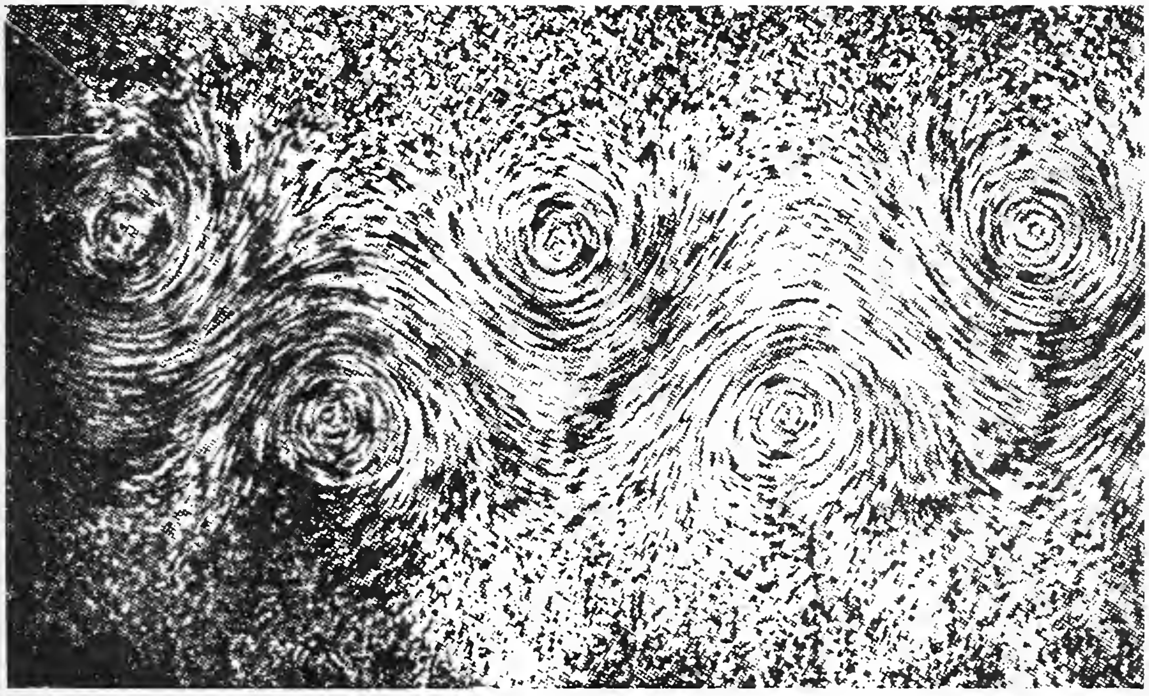

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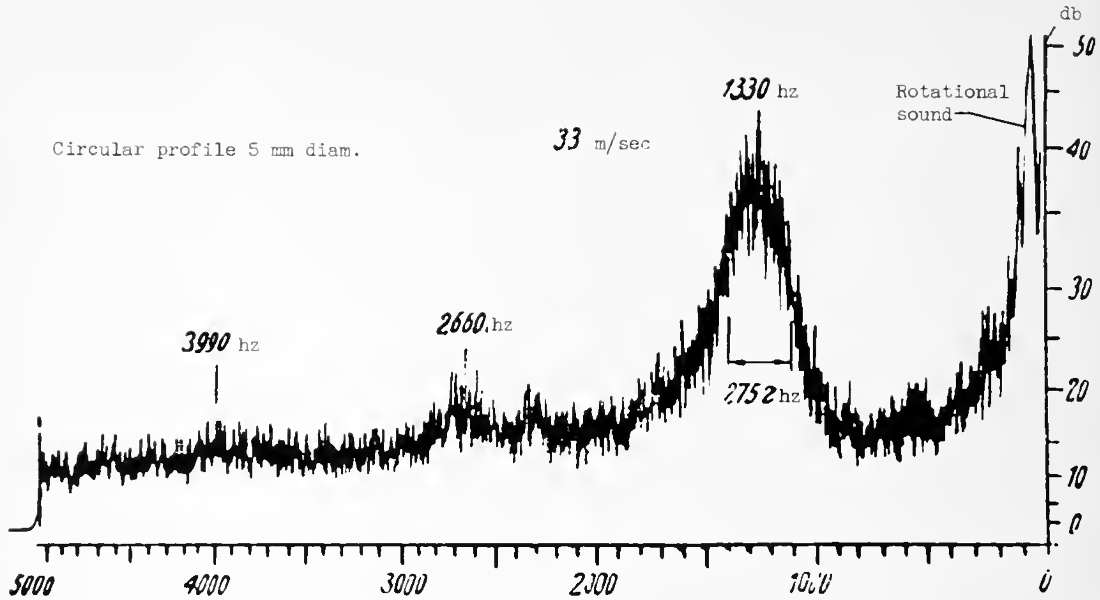

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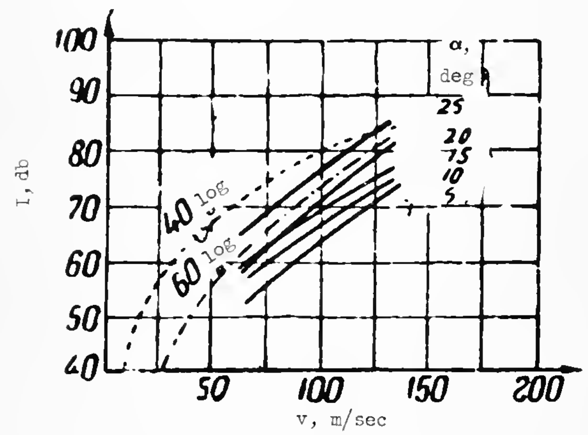

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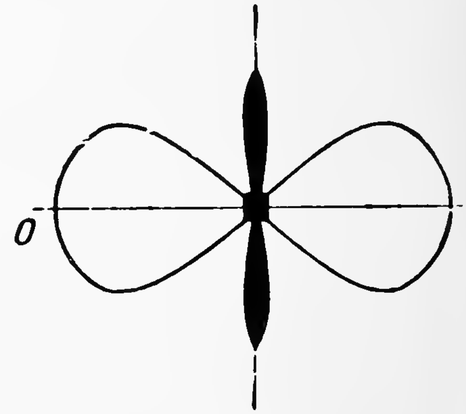

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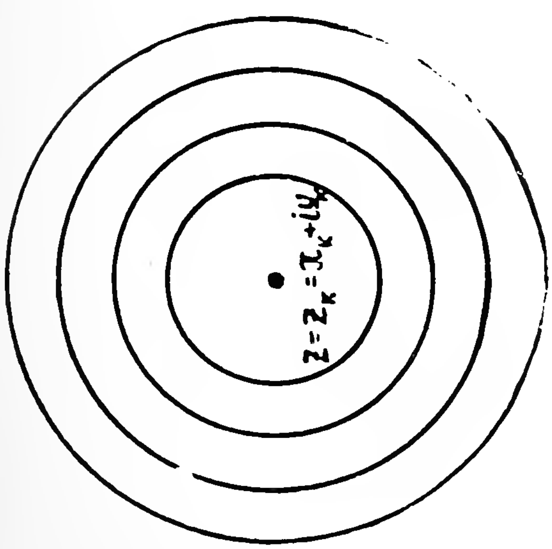

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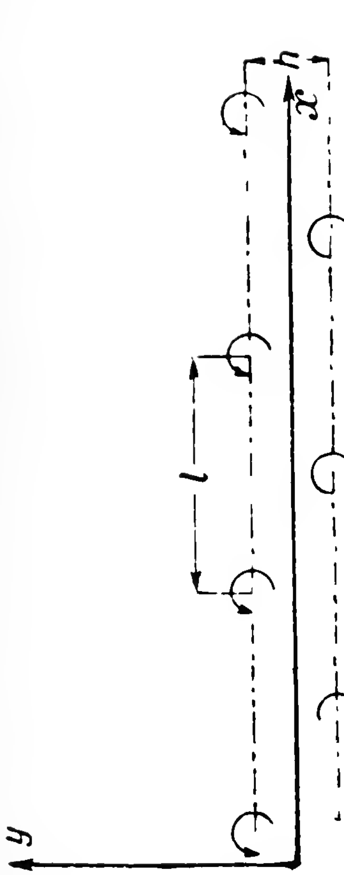

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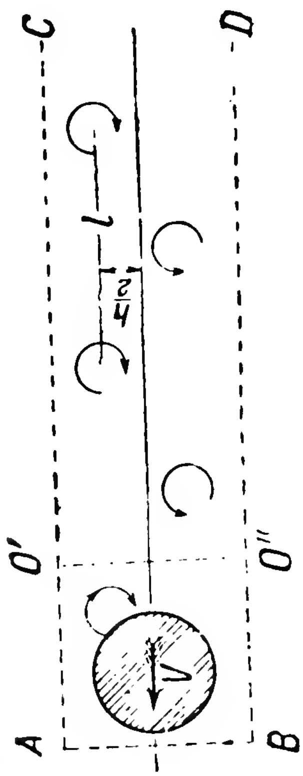

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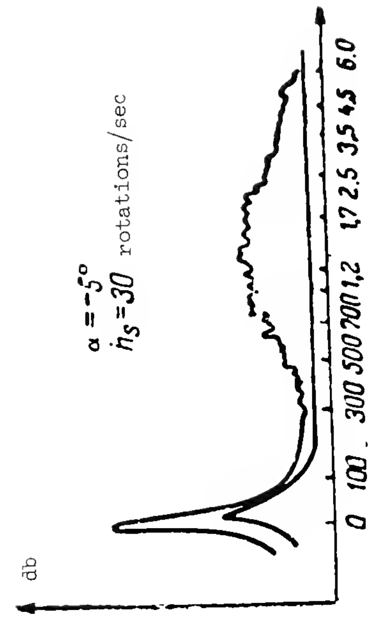

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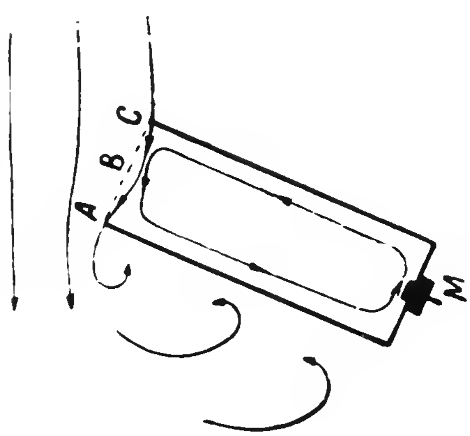

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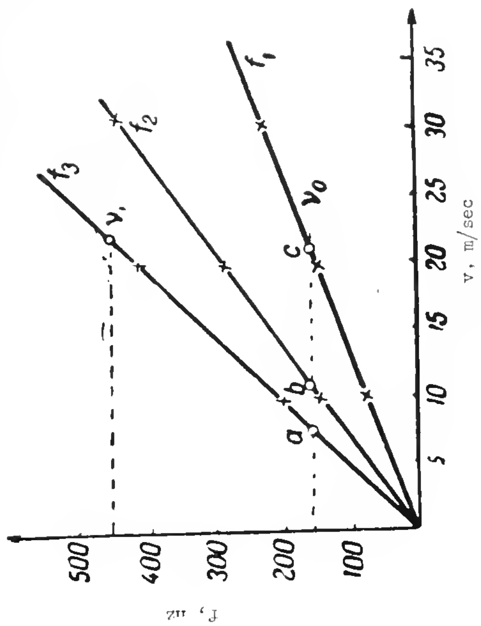

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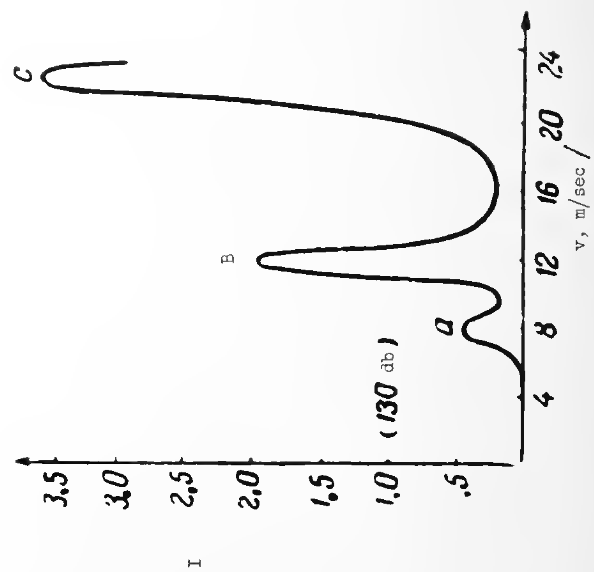

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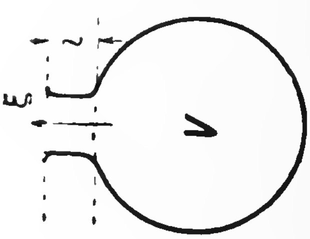

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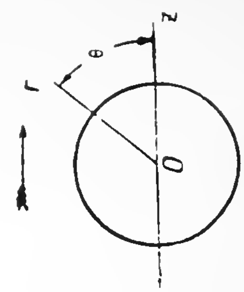

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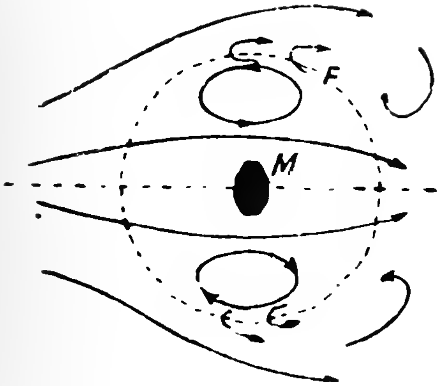

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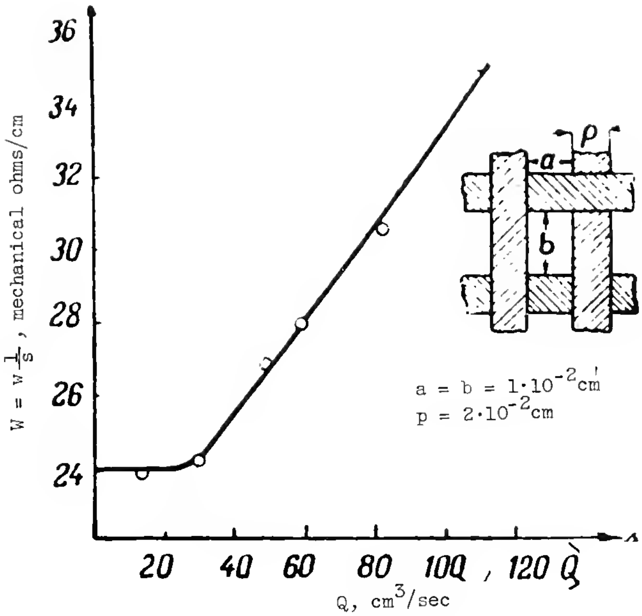

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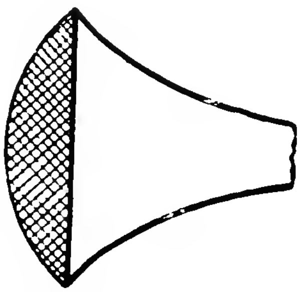

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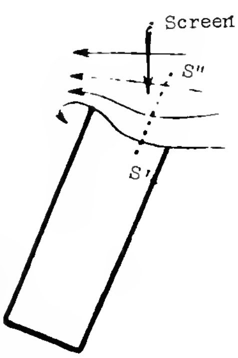

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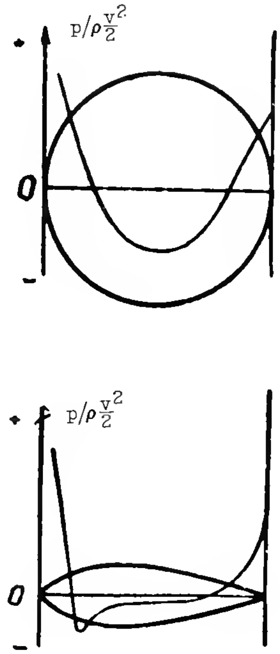

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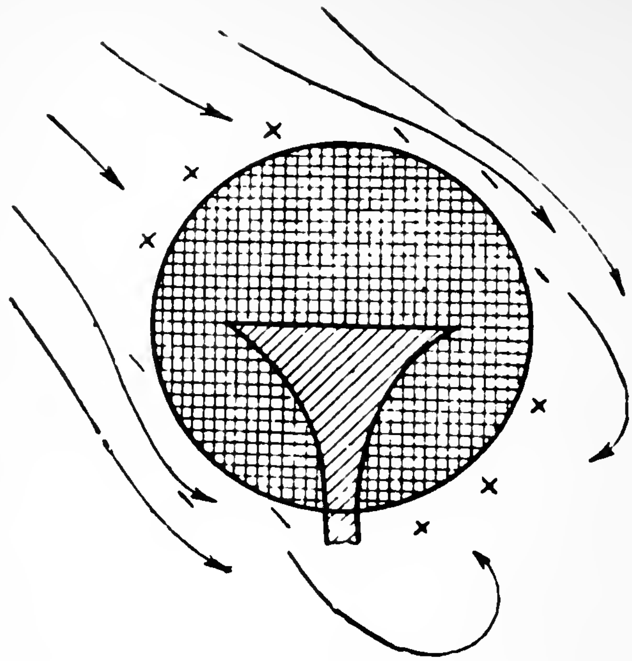

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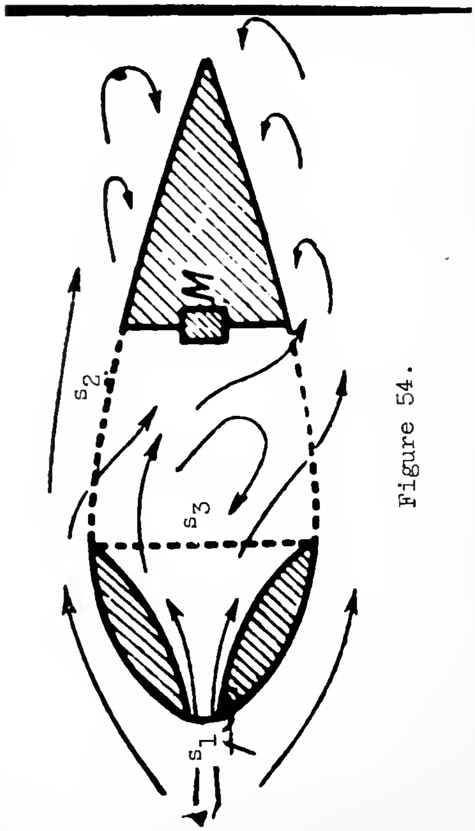

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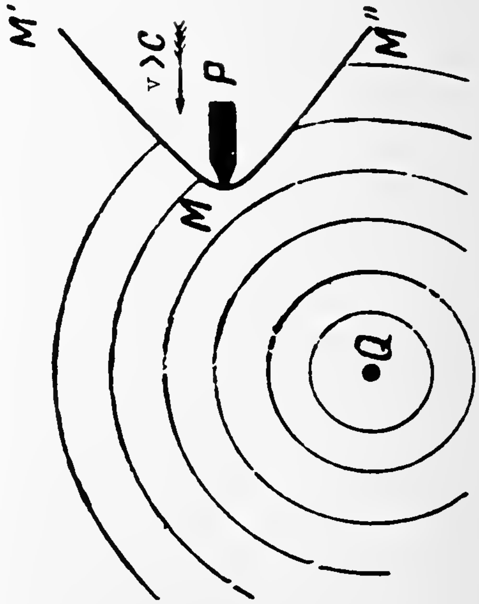

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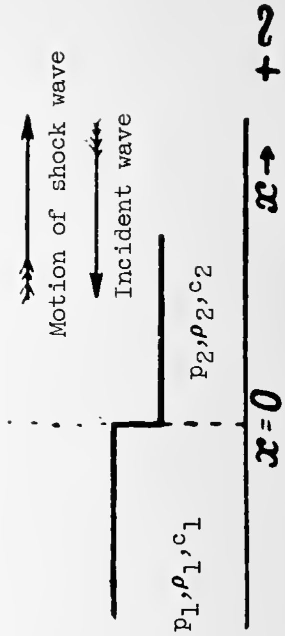

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